

CONSTANT POWER MAPS ON HARDY FIELDS AND TRANSSERIES

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ABSTRACT. Let \mathbb{T} be the differential field of logarithmic-exponential transseries. We consider the expansion of \mathbb{T} by the binary map that sends a real number r and a positive transseries f to the transseries f^r . Building on recent work of Aschenbrenner, van den Dries, and van der Hoeven, we show that this expansion is model complete, and we give an axiomatization of the theory of this expansion that is effective relative to the theory of the real exponential field. We show that maximal Hardy fields, equipped with the same map $(f, r) \mapsto f^r$, enjoy the same theory as \mathbb{T} , and we use this to establish a transfer theorem between Hardy fields and transseries.

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INTRODUCTION

A **Hardy field** H is a field of germs of real-valued unary functions at $+\infty$ that is closed under differentiation: if the germ of f belongs to H , then f must be eventually differentiable and the germ of f' must also belong to H . These axioms rule out any sort of oscillatory behaviour; consequently, any Hardy field is a *totally ordered* differential field.

The quintessential example of a Hardy field is H^{LE} , the field of germs of Hardy's *logarithmico-exponential functions*. These are the functions $f: \mathbb{R} \rightarrow \mathbb{R}$ built from the identity function x by applying exponentials, logarithms, and algebraic operations. This example predates and inspires the general definition, due to Bourbaki. The study of Hardy fields and their Hardy field extensions was greatly advanced in the last quarter of the 20th century by Rosenlicht, Boshernitzan, and others. In the 1990s, Hardy fields became intimately linked with o-minimality, stemming from the observation that an expansion \mathcal{R} of the real field is o-minimal if and only if the collection $H(\mathcal{R})$ of germs of unary \mathcal{R} -definable functions at $+\infty$ forms a Hardy field.

The cardinality of any Hardy field is at most that of the continuum, so any Hardy field is contained in a *maximal* Hardy field by Zorn's lemma. The study of Hardy fields has taken a great leap forward with the recent work of Aschenbrenner, van den Dries, and van der Hoeven, who showed that all maximal Hardy fields have the same first-order theory (as differential fields) [5]. When considered as *valued* differential fields with (existentially definable) valuation ring consisting

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of elements bounded by the germ of a constant function, the theory of maximal Hardy fields is even model complete. This theory can be effectively axiomatized; hence, it is decidable.

This theory has another model of interest—the differential field \mathbb{T} of logarithmic-exponential transseries. This field of generalized power series over \mathbb{R} was introduced independently by Écalle [12] in his solution to the Dulac conjecture and by Dahn and Göring [9] in their work on Tarski’s problem on real exponentiation. The elementary theory of \mathbb{T} (and the elementary theory of maximal Hardy fields) is axiomatized by one of two completions of T^{nl} —the theory of ω -free newtonian Liouville closed H -fields. Introduced by van den Dries and Aschenbrenner [1], H -fields form a class of ordered valued differential fields that includes \mathbb{T} , all differential subfields of \mathbb{T} containing \mathbb{R} , and any Hardy field containing \mathbb{R} . In their book [3], Aschenbrenner, van den Dries, and van der Hoeven proved that the theory T^{nl} is the model companion of the theory of H -fields, gave a quantifier elimination result for this theory in an extended language, and established various model-theoretic properties of this theory. These properties include stable embeddedness of the constant field (as a real closed ordered field) and ω -minimality at $+\infty$. Note that for \mathbb{T} and any Hardy field containing \mathbb{R} , the constant field is exactly the field \mathbb{R} .

Let H be a Hardy field and suppose that H admits an ordered differential field embedding into \mathbb{T} . Examples abound— H^{LE} admits such an embedding, as does $H(\mathbb{R}_{\text{an,exp}})$ [11] and, more generally, $H(\mathbb{R}_{\mathcal{A},\text{exp}})$ for suitable generalized quasianalytic classes \mathcal{A} [23]. Extending the aforementioned results, Aschenbrenner, van den Dries, and van der Hoeven show that any maximal Hardy field extension $M \supseteq H$ has the same first-order theory as \mathbb{T} over H (that is, as a differential field with constant symbols for elements of H , where H is identified with its image in \mathbb{T}). Thus, a system of algebraic differential equations, inequalities, and valuation inequalities over H has a solution in M if and only if it has one in \mathbb{T} [5, Corollary 6].

For the remainder of this introduction, we fix a maximal Hardy field M . Boshernitzan showed that each element in M has an exponential in M , and each positive element has a logarithm in M [7]. Thus, for $f \in M^>$ and $r \in \mathbb{R}$, the germ $f^r := \exp(r \log f)$ belongs to $M^>$. We call the binary map $(f, r) \mapsto f^r : M^> \times \mathbb{R} \rightarrow M^>$ the *constant power map* on M , and we let M_{pow} denote the expansion of the valued differential field M by this map. The transseries also admit a natural exponential, so we similarly obtain a constant power map $(f, r) \mapsto f^r : \mathbb{T}^> \times \mathbb{R} \rightarrow \mathbb{T}^>$. We let \mathbb{T}_{pow} denote the expansion of the valued differential field \mathbb{T} by this map.

Theorem A (Corollaries 5.7 and 6.3). *Both M_{pow} and \mathbb{T}_{pow} have the same first-order theory. This theory is model complete.*

Of course, \mathbb{T}_{pow} is a reduct of \mathbb{T}_{exp} , the expansion of \mathbb{T} by its exponential function. We show that \mathbb{T}_{pow} is a proper reduct in Corollary 5.10 below. Model completeness for \mathbb{T}_{exp} remains open.

To prove Theorem A, we consider H -fields expanded by a map that behaves like the constant power map above. We call these structures *H_{pow} -fields*, and we show that their theory admits a model companion (under some additional assumptions on the field of constants). Both \mathbb{T}_{pow} and M_{pow} are models of this model companion; indeed, this model companion extends the theory of H_{pow} -fields by the same three axioms: ω -freeness, newtonianity, and Liouville closedness. Using this, we can strengthen the first part of Theorem A.

Theorem B (Theorem 6.4). *Let $H \subseteq M_{\text{pow}}$ be a Hardy field closed under the constant power map on M_{pow} , and let $\iota : H \rightarrow \mathbb{T}_{\text{pow}}$ be an ordered differential field embedding that commutes with this map. Then M_{pow} and \mathbb{T}_{pow} have the same first-order theory as differential fields with constant power maps and parameters from H .*

Thus, the transfer theorem for systems of algebraic differential equations over H can be extended to also include *signomial* differential equations (equations involving differential polynomials with real exponents). Both H^{LE} and $H(\mathbb{R}_{\mathcal{A},\text{exp}})$, with their natural embeddings into \mathbb{T} , satisfy the hypotheses on H in the theorem above.

One could, of course, establish a similar transfer theorem by expanding M and \mathbb{T} by unary *real power functions*: functions $f \mapsto f^r$ for each $r \in \mathbb{R}$. As ordered fields with these power functions, M and \mathbb{T} form elementary extensions of the real field with power functions as studied by Miller [21]. This expansion of the real field is o-minimal, and so the corresponding expansions of the valued differential fields M and \mathbb{T} by real power functions are H_T -fields as studied in [18]; see Subsection 5.4. Though model completeness for the expansions of M and \mathbb{T} by real power functions does not follow from our results, it seems fairly easy (though cumbersome) to establish, modifying some of the embedding lemmas here and using the results in [3].

That said, our framework here is more robust, in that it allows for constant powers to be taken *uniformly*. We can use this uniformity to analyze families of signomial ODEs, parametrized by real exponents. As an example of such a family, consider for $\sigma, \lambda, \rho \in \mathbb{R}$ the Emden–Fowler equation

$$\frac{d}{dx} \left(x^\sigma \frac{dy}{dx} \right) + x^\rho y^\lambda = 0. \quad (\text{EF}_{\sigma, \rho, \lambda})$$

The asymptotic behavior of solutions to this equation for various values of the three real parameters was studied by Bellman [6]. This equation is definable over H^{LE} , so Theorem B tells us that $(\text{EF}_{\sigma, \rho, \lambda})$ has a positive Hardy field solution $y(x)$ if and only if it has a positive solution in \mathbb{T} . If we are interested in the structure of the set

$$\{(\sigma, \rho, \lambda) \in \mathbb{R}^3 : (\text{EF}_{\sigma, \rho, \lambda}) \text{ has a positive solution in } \mathbb{T}\},$$

we need to know what structure is induced on the constant field \mathbb{R} . We note that \mathbb{R} is no longer stably embedded as a pure field, as the map $r \mapsto 2^r : \mathbb{R} \rightarrow \mathbb{R}$ is definable in \mathbb{T}_{pow} . This turns out to be the only new structure:

Theorem C (Corollary 5.12). *In \mathbb{T}_{pow} , the real numbers are stably embedded as a real exponential field: any definable subset of \mathbb{R}^n is definable in the structure \mathbb{R}_{exp} .*

The structure \mathbb{R}_{exp} was shown to be model complete and o-minimal by Wilkie [25]. Given a family of signomial ODEs parametrized by real exponents, like the family $(\text{EF}_{\sigma, \rho, \lambda})$ above, it follows that whether the equations in this family have a transseries or Hardy field solution only depends on the parameters in a geometrically tame way—e.g., it is never dependent on rationality or algebraicity of the exponents.

The fact that the exponential on \mathbb{R} is definable in \mathbb{T}_{pow} tells us that the theory of this structure is at least as complicated as that of \mathbb{R}_{exp} . This is the only additional complication.

Theorem D (Corollary 5.6). *The theory of \mathbb{T}_{pow} is decidable relative to the theory of \mathbb{R}_{exp} .*

Decidability for the theory of \mathbb{R}_{exp} is intricately linked with transcendental number theory (it is implied by Schanuel’s conjecture; see [20]). Of course, Theorems C and D hold for M_{pow} as well, by Theorem A.

At the end of the paper, we also study Conway’s field of surreal numbers, equipped with the Berarducci–Mantova derivation and the natural constant power map induced by the Gonshor–Kruskal exponential. Building on results from [4], we show that this expansion of the surreals, denoted \mathbf{No}_{pow} , has the same first-order theory as \mathbb{T}_{pow} , and that every Hardy field closed under constant powers admits an embedding into the surreals.

Outline. We collect necessary preliminary results, mostly from the book [3], in Section 1. In Section 2, we investigate power extensions and power closures of H -fields; much of the material needed here comes from [2, Sections 7 and 8]. In Section 3, we introduce H_{pow} -fields, our primary objects of study. The focus of this section is our central technical result: Proposition 3.4. This proposition allows us to convert facts from [3] on H -field embeddings and extensions to corresponding results about H_{pow} -fields. We apply this proposition throughout Section 4, where we collect the embedding results required for our main theorems.

In Section 5, we prove that the theory of H_{pow} -fields has a model companion, characterize the completions of this model companion, and show that \mathbb{T}_{pow} is a model. We also study definable sets in the model companion, proving stable embeddedness of the constant field, local o-minimality, and non-definability of the exponential on \mathbb{T} . In Section 6, we prove our results on maximal Hardy fields and the surreal numbers, including Theorem B.

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1. PRELIMINARIES

We draw heavily from [3], and we assume some familiarity with that book. We use the same notational conventions as [3], but will repeat what is needed in this paper.

We let m , n , and k range over $\mathbb{N} = \{0, 1, 2, \dots\}$. For a ring R , we let R^\times denote the invertible elements of R . By “ordered set” we mean “totally ordered set”. For an ordered abelian group Γ , we put

$$\Gamma^> := \{\gamma \in \Gamma : \gamma > 0\}, \quad \Gamma^< := \{\gamma \in \Gamma : \gamma < 0\}.$$

Given a (possibly infinite) index set I , we say that a tuple $(\gamma_i)_{i \in I}$ of elements from Γ is **finitely supported** if the set $\{i \in I : \gamma_i \neq 0\}$ is finite. We let $\mathbb{Q}\Gamma := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ denote the divisible hull of Γ , equipped with the unique ordering that makes $\mathbb{Q}\Gamma$ an ordered group extension of Γ . For $S \subseteq \Gamma$, we set

$$S^\downarrow := \{\gamma \in \Gamma : \gamma \leq \sigma \text{ for some } \sigma \in S\}.$$

1.1. Ordered exponential fields, valued fields, and differential fields. In this subsection, let K be a field of characteristic zero.

Ordered exponential fields. Let $<$ be a field ordering on K . An **exponential** on K is an ordered group isomorphism $\exp: K \rightarrow K^>$ from the additive group of K to the multiplicative group of positive elements of K . If K and L are ordered exponential fields (that is, ordered fields equipped with exponentials), then an ordered field embedding $\iota: K \rightarrow L$ is said to be an **ordered exponential field embedding** if $\iota(\exp a) = \exp \iota(a)$ for all $a \in K$. We let \mathbb{R}_{exp} denote the real ordered exponential field. This structure is o-minimal, and its first-order theory $\text{Th}(\mathbb{R}_{\text{exp}})$ is model complete in the natural language $\{0, 1, +, -, \cdot, <, \exp\}$ of ordered exponential fields by Wilkie’s theorem [25].

Valued fields. Let $\mathcal{O} \subseteq K$ be a valuation ring on K , so each element of K either belongs to \mathcal{O} or has a multiplicative inverse in \mathcal{O} . We denote the unique maximal ideal of \mathcal{O} by \mathfrak{o} , the residue field \mathcal{O}/\mathfrak{o} by $\text{res}(K)$, the value group of K (written additively) by Γ , and the surjective valuation map $K^\times \rightarrow \Gamma$ by v . The group Γ is totally ordered, where $va \geq 0 \iff a \in \mathcal{O}$ and $va > 0 \iff a \in \mathfrak{o}$. We extend v to a map $K \rightarrow \Gamma_\infty := \Gamma \cup \{\infty\}$ by setting $v(0) := \infty > \Gamma$. For $a, b \in K$ we have the following notations:

$$a \asymp b : \iff va = vb, \quad a \preceq b : \iff va \geq vb, \quad a \prec b : \iff va > vb, \quad a \sim b : \iff a - b \prec a.$$

Note that $a \sim b$ if and only if $a, b \neq 0$ and $a/b \in 1 + \mathfrak{o}$. If L is also a valued field, then we denote its value group, valuation ring, and maximal ideal by Γ_L , \mathcal{O}_L , and \mathfrak{o}_L , respectively. Let L be a valued field extension of K . We identify Γ with a subgroup of Γ_L and $\text{res}(K)$ with a subfield of $\text{res}(L) = \mathcal{O}_L/\mathfrak{o}_L$ in the natural way. We say that L is an **immediate extension of K** if $\Gamma_L = \Gamma$ and $\text{res}(L) = \text{res}(K)$ under this identification.

Differential fields. Let $\partial: K \rightarrow K$ be a derivation on K , so $\partial(a+b) = \partial a + \partial b$ and $\partial(ab) = a\partial b + b\partial a$. We let $C := \ker(\partial)$ denote the constant field of K . For $a \in K$, we often write a' instead of ∂a , and if $a \neq 0$, we let $a^\dagger := a'/a$ denote the logarithmic derivative of a . We set $(K^\times)^\dagger := \{a^\dagger : a \in K^\times\}$. Then $(K^\times)^\dagger$ is a subgroup of K , since $(a^{-1})^\dagger = -a^\dagger$ and $a^\dagger + b^\dagger = (ab)^\dagger$ for $a, b \in K^\times$. An **integral** of $a \in K$ is an element $f \in K$ with $f' = a$, and an **exponential integral** of a is an element $g \in K^\times$ with $g^\dagger = a$. If f_1, f_2 are integrals of a and g_1, g_2 are exponential integrals of a , then $f_1 - f_2 \in C$ and $g_1/g_2 \in C^\times$.

If L is also a differential field, then we denote its constant field by C_L . If y is an element of a differential field extension of K , then let $K\langle y \rangle := K(y, y', y'', \dots)$ denote the differential field extension of K generated by y . We say that y is **d-transcendental over K** if the sequence y, y', y'', \dots is algebraically independent over K , and we say that y is **d-algebraic over K** otherwise.

1.2. H -fields. Let K be an ordered differential field with constant field C . We let \mathcal{O} be the convex hull of C in K , and we view K as an ordered valued differential field with valuation ring \mathcal{O} .

We say that K is an **H -field** if

(H1) $f' > 0$ for all $f \in K$ with $f > \mathcal{O}$;

(H2) $\mathcal{O} = C + \mathfrak{o}$.

The class of H -fields was introduced by Aschenbrenner and van den Dries [1]. Any ordered field with trivial derivation is a trivially valued H -field, and every H -field with nontrivial derivation has a nontrivial valuation ring.

For the remainder of this section, let K be an H -field. An **H -field embedding** is an ordered valued differential field embedding of K into an H -field L . By (H2), the projection map $\mathcal{O} \rightarrow \text{res}(K)$ maps C isomorphically onto $\text{res}(K)$. Consequently, an H -field extension L of K is an immediate extension of K if and only if $\Gamma_L = \Gamma$ and $C_L = C$. By (H1), we have for $f \in K^>$ that

$$f \succ 1 \implies f^\dagger > 0, \quad f \prec 1 \implies f^\dagger = -(f^{-1})^\dagger < 0.$$

Lemma 1.1. *Let $s \in K$, let M_1, M_2 be H -field extensions of K , and for $i = 1, 2$, let $f_i \in M_i^>$ with $f_i^\dagger = s$ and $vf_i \notin \mathbb{Q}\Gamma$. Then there is a unique H -field embedding $K(f_1) \rightarrow M_2$ over K that sends f_1 to f_2 .*

Proof. As $vf_i \notin \mathbb{Q}\Gamma$, each f_i is transcendental over K , so there is a field embedding $K(f_1) \rightarrow M_2$ over K that sends f_1 to f_2 . This is a differential field embedding, as $f_1^\dagger = f_2^\dagger$. To see that this is an H -field embedding, it is enough to show that vf_1 and vf_2 realize the same cut over $\mathbb{Q}\Gamma$. To see this, let $m, n \in \mathbb{N}$ with $n > 0$, let $\gamma \in \Gamma$, and suppose that $nvf_1 < m\gamma$ but $m\gamma < nvf_2$. Take $y \in K^>$ with $vy = \gamma$, so $f_1^n/y^m \succ 1$ and $f_2^n/y^m \prec 1$. As $f_1, f_2, y > 0$, this gives $(f_1^n/y^m)^\dagger = ns - my^\dagger > 0$, whereas $(f_2^n/y^m)^\dagger = ns - my^\dagger < 0$, a contradiction. \square

1.3. Asymptotic couples and the trichotomy theorem. As an H -field, K is **H -asymptotic**: for all $f, g \in \mathfrak{o}$, we have

$$f \prec g \iff f' \prec g', \quad f \preceq g \implies f^\dagger \succeq g^\dagger$$

It follows that for $f \in K^\times$ with $f \neq 1$, the values $v(f')$ and $v(f^\dagger)$ only depend on vf , so for $\gamma = vf$, we set

$$\gamma^\dagger := v(f^\dagger), \quad \gamma' := v(f') = \gamma + \gamma^\dagger.$$

This gives us a map

$$\psi: \Gamma^\neq \rightarrow \Gamma, \quad \psi(\gamma) := \gamma^\dagger$$

and, following Rosenlicht [24], we call the pair (Γ, ψ) the **asymptotic couple of K** . We have the following important subsets of Γ :

$$(\Gamma^<)' := \{\gamma' : \gamma \in \Gamma^<\}, \quad (\Gamma^>)' := \{\gamma' : \gamma \in \Gamma^>\}, \quad \Psi := \{\gamma^\dagger : \gamma \in \Gamma^\neq\}.$$

It is always the case that $(\Gamma^<)' < (\Gamma^>)'$ and that $\Psi < (\Gamma^>)'$. For $u \in K$ with $u \asymp 1$, there is $c \in C$ with $u - c \prec 1$, so $v(u') = v(u - c)' \in (\Gamma^>)'$. then $v(u^\dagger) \in (\Gamma^>)'$ as well, since $u^\dagger = u'/u \asymp u'$. We say that K has **small derivation** if $\partial\mathcal{O} \subseteq \mathcal{O}$ (equivalently, if $(\Gamma^>)' \subseteq \Gamma$). If this is not the case, then K is said to have **large derivation**. Note that if K has large derivation if and only if $0 \in (\Gamma^>)'$.

If there is $\beta \in \Gamma$ with $\Psi < \beta < (\Gamma^>)'$, then we call β a **gap in K** . There is at most one such β , and if Ψ has a largest element, then there is no such β . If K has trivial valuation (equivalently, if K has trivial derivation, since K is an H -field), then the three subsets above are empty, and 0 is a gap in K . We say that K is **grounded** if Ψ has a largest element, and we say that K is **ungrounded** otherwise. Finally, we say that K has **asymptotic integration** if $\Gamma = (\Gamma^<)' \cup (\Gamma^>)'$. If β is a gap in K or if $\beta = \max \Psi$, then $\Gamma = (\Gamma^<)' \cup \{\beta\} \cup (\Gamma^>)'$. We have an important trichotomy for the structure of H -asymptotic fields:

Fact 1.2 ([3, Corollary 9.2.16]). *If K is an H -asymptotic field, then exactly one of the following is true:*

- (1) K has asymptotic integration;
- (2) K has a gap;
- (3) K is grounded.

1.4. ω -freeness, newtonianity and d-algebraic maximality. A major result in [3] is that the theory of H -fields has a model companion, namely the theory T^{ml} of ω -free newtonian Liouville closed H -fields. The H -field K is **Liouville closed** if K is real closed and for each $s \in K$, there is $a \in K$ and $f \in K^\times$ with $a' = f^\dagger = s$. The axioms “ ω -free” and “newtonian” are more technical, and we will not define these axioms precisely, but we will list some facts about these axioms that will be useful later in this paper.

The property of **ω -freeness** is a rather subtle axiom that, among other things, rules out the existence of gaps both in K and in various extensions of K . Every ω -free H -field has asymptotic integration. This property is also quite robust; it passes to d-algebraic H -field extensions, and it is inherited by certain H -subfields:

Fact 1.3 ([3, Section 11.7 and Theorem 13.6.1]). *Suppose that K is ω -free. If L is a d-algebraic H -field extension of K , then L is ω -free. If E is an H -subfield of K with $\Gamma_E^<$ cofinal in $\Gamma^<$, then E is ω -free.*

In connection with gaps, let us mention another consequence of ω -freeness:

Lemma 1.4. *Let K be ω -free and let f be an element in an H -field extension of K such that vf is a gap in $K\langle f \rangle$. Then $K\langle f^\dagger \rangle$ is an immediate extension of K .*

Proof. This follows from [3, Lemmas 11.4.7, 11.5.6, and 11.5.9]. For readers who are familiar with λ -sequences from [3], we provide a proof. If vf is a gap in $K\langle f \rangle$, then $-f^\dagger$ is a pseudolimit of a λ -sequence in K by [3, Lemmas 11.5.6 and 11.5.9], but since K is ω -free, any λ -sequence in K is divergent and of d-transcendental type over K [3, Corollary 13.6.3], so using [3, Lemma 11.4.7], we get that $K\langle f^\dagger \rangle$ is an immediate extension of K . \square

The axiom of newtonianity is also rather subtle. It is, however, quite a strong axiom, especially when coupled with ω -freeness:

Fact 1.5. *Let K be an ω -free newtonian H -field. Then K is **asymptotically d-algebraically maximal**, that is, K has no proper immediate d-algebraic H -field extensions.*

Fact 1.5 was shown under the assumption that K also has divisible value group [3, Theorem 14.0.2], but this divisibility assumption can be removed; see [22].

Lemma 1.6. *Suppose that K is ω -free and newtonian and let $s \in K$. Then there is $a \in K$ with $a' = s$, and if $vs > \Psi$, then there is $f \in K$ with $f^\dagger = s$.*

Proof. By [3, Proposition 10.2.6], we can find an element a in an immediate H -field extension of K with $a' = s$. As K is asymptotically d -algebraically maximal, a belongs to K . Now suppose that $vs > \Psi$. As K has asymptotic integration, $vs \in (\Gamma^{>})'$, so [3, Lemma 10.4.3] gives f in an immediate H -field extension of K with $f \neq 0$ and $f^\dagger = s$. Again, asymptotic d -algebraic maximality gives $f \in K$. \square

If K is ω -free and newtonian, then by Fact 1.5, any y in an immediate H -field extension of K must be d -transcendental over K . By [3, Lemmas 2.4.2 and 11.4.7], together with other facts from [3, Section 11.4], the H -field extension $K\langle y \rangle$ is completely determined by the cut of y over K :

Fact 1.7. *Let K be ω -free and newtonian and let $K\langle y \rangle$ be an immediate H -field extension of K . Let L be an H -field extension of K and let $z \in L$ realize the same cut as y over K . Then there is an H -field embedding $K\langle y \rangle \rightarrow L$ over K that sends y to z .*

If K is ω -free, then a **newtonization of K** is by definition an immediate newtonian H -field extension of K that embeds over K into any newtonian H -field extension of K .

Fact 1.8. *If K is ω -free, then K has a newtonization K^{nt} that is d -algebraic over K . Any two newtonizations of K are isomorphic over K .*

Fact 1.8 was shown under the assumption that Γ is divisible [3, Corollaries 14.3.12 and 14.5.4] but again, this divisibility assumption can be removed by [22].

When combined with Liouville closedness, ω -freeness and newtonianity give d -algebraic maximality up to constant field extensions:

Fact 1.9 ([3, Theorem 16.0.3]). *Let K be an ω -free newtonian Liouville closed H -field. Then K has no proper d -algebraic H -field extensions with the same constant field.*

If K is ω -free, then a **Newton-Liouville closure of K** is by definition a newtonian Liouville closed H -field extension of K that embeds over K into any newtonian Liouville closed H -field extension of K .

Fact 1.10 ([3, Corollaries 14.5.10 and 16.2.2]). *Suppose that K is ω -free. Then K has a Newton-Liouville closure K^{nl} that is d -algebraic over K . The constant field of K^{nl} is a real closure of C , and any other Newton-Liouville closure of K is isomorphic to K^{nl} over K .*

Combining the previous two facts, we can characterize the Newton-Liouville closures of K as follows:

Lemma 1.11. *Let K be ω -free and let L be a newtonian Liouville closed H -field extension of K . Then L is a Newton-Liouville closure of K if and only if L is d -algebraic over K and C_L is algebraic over C .*

Proof. One direction follows immediately from Fact 1.10. For the other, suppose that L is d -algebraic over K and C_L is algebraic over C , and let K^{nl} be a Newton-Liouville closure of K . We may identify K^{nl} with a subfield of L ; then L is a d -algebraic extension of K^{nl} and C_L , being the real closure of C , coincides with the constant field of K^{nl} . Fact 1.9 gives $L = K^{\text{nl}}$. \square

This gives us some useful cardinality bounds:

Corollary 1.12. *Let K be ω -free. Then any Newton-Liouville closure of K has the same cardinality as K .*

Proof. Let K^{nl} be a Newton-Liouville closure of K . The downward Löwenheim–Skolem theorem gives an elementary H -subfield $L \subseteq K^{\text{nl}}$ that contains K with $|L| = |K|$. Then K^{nl} is a d -algebraic H -field extension of L with the same constant field as L , so $L = K^{\text{nl}}$ by Fact 1.9. \square

Corollary 1.13. *Let K be ω -free and let L be a d -algebraic H -field extension of K . If C_L is algebraic over C , then $|L| = |K|$.*

Proof. By Fact 1.3, L is ω -free as well. Let L^{nl} be a Newton-Liouville closure of L . Then L^{nl} is d -algebraic over K and $C_{L^{\text{nl}}}$ is algebraic over C , so Lemma 1.11 gives that L^{nl} is a Newton-Liouville closure of K . Corollary 1.12 gives $|K| \leq |L| \leq |L^{\text{nl}}| = |K|$. \square

2. POWER EXTENSIONS AND THE POWER CLOSURE

In this section, let K be an H -field with real closed constant field C . Following [2], we say that K is **closed under powers** if for all $f \in K^\times$ and all $c \in C$, there is $g \in K^\times$ with $g^\dagger = cf^\dagger$. Equivalently, K is closed under powers if the additive subgroup $(K^\times)^\dagger \subseteq K$ is a C -linear subspace of K .

Fact 2.1 ([2, Lemma 7.4]). *Suppose that K is closed under powers. Then Γ admits the structure of an ordered C -vector space. Explicitly, for $\gamma \in \Gamma$ and $c \in C$, take $f, g \in K^\times$ with $vf = \gamma$ and $g^\dagger = cf^\dagger$ and set $c\gamma := vg$. This choice does not depend on our choice of f or g .*

For $g \in K^\times$, we have $g \asymp 1$ if and only if $v(g^\dagger) > \Psi$. This gives us the following:

Lemma 2.2. *Suppose that K is closed under powers, let $f_1, \dots, f_n \in K^\times$, and let $c_1, \dots, c_n \in C$. Then*

$$c_1vf_1 + \dots + c_nv f_n = 0 \iff v(c_1f_1^\dagger + \dots + c_nf_n^\dagger) > \Psi.$$

Proof. Choose for each $i = 1, \dots, n$ some $g_i \in K$ with $g_i^\dagger = c_if_i^\dagger$ and set $g := g_1g_2 \dots g_n$. Then

$$v(c_1f_1^\dagger + \dots + c_nf_n^\dagger) = v(g^\dagger) > \Psi \iff c_1vf_1 + \dots + c_nv f_n = vg = 0. \quad \square$$

Lemma 2.3. *Suppose that K is closed under powers, let $f \in 1 + \mathfrak{o}$, and let $c \in C$. Then there is a unique $g \in 1 + \mathfrak{o}$ with $g^\dagger = cf^\dagger$.*

Proof. Take $a \in K^\times$ with $a^\dagger = cf^\dagger$. Then $a \asymp 1$, so we may take $d \in C^\times$ with $a \sim d$. Put $g := a/d$. Then $g \sim 1$ and $g^\dagger = a^\dagger = cf^\dagger$. To see that g is unique, let $h \in 1 + \mathfrak{o}$ with $h^\dagger = cf^\dagger = g^\dagger$. Then

$$h/g \in C^\times \cap (1 + \mathfrak{o}) = \{1\},$$

so $h = g$. \square

A **power extension** of K is an H -field extension L of K such that $C_L = C$ and each $a \in L$ is contained in an intermediate subfield $K(t_1, \dots, t_n) \subseteq L$ where for each i , either t_i is algebraic over $K(t_1, \dots, t_{i-1})$ or $t_i^\dagger = cf^\dagger$ for some $c \in C$ and some nonzero $f \in K(t_1, \dots, t_{i-1})$. A **power closure** of K is a power extension of K that is real closed and closed under powers. In [2, Section 8], Aschenbrenner and van den Dries studied power closures, establishing the following:

Fact 2.4 ([2, Proposition 8.5]). *K has either exactly one or exactly two power closures up to K -isomorphism.*

Moreover, they characterized when K has two distinct power closures:

Fact 2.5 ([2, Proposition 8.10]). *K has two power closures nonisomorphic over K if and only if there are $c_1, \dots, c_n \in C$ and $f_1, \dots, f_n \in K^\times$ such that*

$$\Psi < v(c_1f_1^\dagger + \dots + c_nf_n^\dagger) < (\Gamma^>)'.$$

In particular, if K is grounded or has asymptotic integration, then K has a unique power closure.

If L is a power closure of K , then Γ_L admits the structure of an ordered C -vector space, so it makes sense to speak of $C\Gamma$, the C -linear span of Γ , as a subspace of Γ_L . In the case that K has a unique power closure, we have a good understanding of the value group and Ψ -set of this power closure.

Fact 2.6 ([2, Corollary 8.11]). *Suppose that K has a unique power closure K_{pow} . Then $\Gamma_{K_{\text{pow}}} = C\Gamma$. If K is grounded, then $\max \Psi_{K_{\text{pow}}} = \max \Psi$, and if K is ungrounded, then $\Gamma^>$ is coinitial in $\Gamma_{K_{\text{pow}}}^>$. Consequently, Ψ is cofinal in $\Psi_{K_{\text{pow}}}$ and a gap in Γ remains a gap in $\Gamma_{K_{\text{pow}}}$.*

Lemma 2.7. *Suppose that K has a unique power closure K_{pow} and let $M \subseteq K_{\text{pow}}$ be an H -field extension of K . Then K_{pow} is also the unique power closure of M .*

Proof. Clearly, K_{pow} is a power closure of M , as any H -field extension of M that is a power extension of K is also a power extension of M . Thus, we need only show that M has a unique power closure. If K is grounded, then this follows from Facts 2.5 and 2.6, since $\Psi \subseteq \Psi_M$ is cofinal in $\Psi_{K_{\text{pow}}}$. Suppose that K is ungrounded, so $\Gamma^> \subseteq \Gamma_M^>$ is coinitial in $\Gamma_{K_{\text{pow}}}^>$. Let $f_1, \dots, f_n \in M^>$ and $c_1, \dots, c_n \in C$ and take $g \in K_{\text{pow}}$ with $g^\dagger = c_1 f_1^\dagger + \dots + c_n f_n^\dagger$. Either $vg^\dagger \in \Psi_{K_{\text{pow}}}$ or $vg = 0$ and $vg^\dagger = vg' \in (\Gamma_{K_{\text{pow}}}^>)'$. Since Ψ_M is cofinal in $\Psi_{K_{\text{pow}}}$ and $(\Gamma_M^>)'$ is coinitial in $(\Gamma_{K_{\text{pow}}}^>)'$, we can conclude by Fact 2.5 that M has a unique power closure. \square

It may well happen that K has a unique power closure up to isomorphism but not up to *unique* isomorphism. However, in the case that K_{pow} is an immediate extension of K (or, more generally, of the real closure of K), then this isomorphism is indeed unique.

Lemma 2.8. *Suppose that K has a unique power closure K_{pow} and suppose that $\mathbb{Q}\Gamma = \Gamma_{K_{\text{pow}}}$. Then K_{pow} is unique up to unique K -isomorphism: if M is an H -field extension of K that is closed under powers, then there is a unique H -field embedding $K_{\text{pow}} \rightarrow M$ over K .*

Proof. Let $\iota, j: K_{\text{pow}} \rightarrow M$ be two H -field embeddings over K . We need to show that $\iota = j$. Using Zorn's lemma, we let $K_0 \subseteq K_{\text{pow}}$ be the maximal subset of K_{pow} on which ι and j agree, so K_0 is an H -field extension of K . Then K_0 is real closed, as ι and j extend uniquely to the real closure of K_0 . Thus, $\Gamma_{K_0} = \Gamma_{K_{\text{pow}}}$, so K_{pow} is an immediate extension of K_0 .

Let $f \in K_{\text{pow}}^\times$ with $f^\dagger \in K_0$. Then $\iota(f^\dagger) = j(f^\dagger)$, so $\iota(f)/j(f) \in C_M^\times$. Take $g \in K_0^\times$ with $f \sim g$, and put $u := f/g \sim 1$. We have $\iota(g) = j(g)$, so

$$\iota(f)/j(f) = \iota(u)/j(u) \in C_M^\times$$

As $\iota(u) \sim j(u) \sim 1$, we conclude that $\iota(u)/j(u) = 1$, so $\iota(f) = j(f)$ and $f \in K_0$. Thus, K_0 has no proper power extensions in K_{pow} , so $K_0 = K_{\text{pow}}$. \square

2.1. Linear independence in H -fields. As evidenced by the criterion in Fact 2.5, C -linear combinations of elements of K play a crucial role in determining the structure of power extensions of K . In the proof of our main technical result, Proposition 3.4, we will need a handful of lemmas about these C -linear combinations, which we include below.

Lemma 2.9. *Let $f_1, \dots, f_n \in K^\times$ and $c_1, \dots, c_n \in C$. If c_n is in the \mathbb{Q} -linear span of $\{c_1, \dots, c_{n-1}\}$, then we can find $g_1, \dots, g_{n-1} \in K^\times$ with*

$$c_1 f_1^\dagger + \dots + c_n f_n^\dagger \asymp c_1 g_1^\dagger + \dots + c_{n-1} g_{n-1}^\dagger.$$

Proof. Take $m_1, \dots, m_n \in \mathbb{Z}$ with $m_n \neq 0$ and $m_n c_n = m_1 c_1 + \dots + m_{n-1} c_{n-1}$. For each $j = 1, \dots, n-1$, set $g_j := f_j^{m_n} f_n^{m_j}$, so we have

$$\begin{aligned} c_1 g_1^\dagger + \dots + c_{n-1} g_{n-1}^\dagger &= c_1 (m_n f_1^\dagger + m_1 f_n^\dagger) + \dots + c_{n-1} (m_n f_{n-1}^\dagger + m_{n-1} f_n^\dagger) \\ &= m_n c_1 f_1^\dagger + \dots + m_n c_{n-1} f_{n-1}^\dagger + (m_1 c_1 + \dots + m_{n-1} c_{n-1}) f_n^\dagger \\ &= m_n (c_1 f_1^\dagger + \dots + c_n f_n^\dagger) \asymp c_1 f_1^\dagger + \dots + c_n f_n^\dagger. \end{aligned} \quad \square$$

By the axiom (H2), the underlying valued C -vector space of K is actually a Hahn space over K ; see [3, Section 2.3]. The next Lemma follows from [3, Lemmas 2.3.1 and 4.6.16], but we include a proof for completeness.

Lemma 2.10. *Let M be an H -field extension of K and let $f \in K[C_M] \subseteq M$. Then $f = d_1g_1 + \cdots + d_mg_m$ for some $d_1, \dots, d_m \in C_M$ and $g_1, \dots, g_m \in K^\times$ with vg_1, \dots, vg_m distinct.*

Proof. Write $f = c_1f_1 + \cdots + c_nf_n$, with $c_1, \dots, c_n \in C_M$ and $f_1, \dots, f_n \in K$. We will prove the lemma by induction on n , with the case $n = 0$ holding trivially. Suppose $n > 0$ and, applying the lemma to $c_1f_1 + \cdots + c_{n-1}f_{n-1}$, take $d_1, \dots, d_m \in C_M$ and $g_1, \dots, g_m \in K^\times$ with distinct valuations such that

$$c_1f_1 + \cdots + c_{n-1}f_{n-1} = d_1g_1 + \cdots + d_mg_m.$$

We claim that we can find $\tilde{c}_1, \dots, \tilde{c}_m \in C$ with

$$v(f_n - \tilde{c}_1g_1 - \cdots - \tilde{c}_mg_m) \notin \{vg_1, \dots, vg_m\}.$$

If $vf_n \notin \{vg_1, \dots, vg_m\}$, we may take all the \tilde{c}_i to be zero; otherwise, use that if $f_n \succ g_i$, then $f_n - \tilde{c}g_i \prec g_i$ for some $\tilde{c} \in C$ by (H2). Put $h := f_n - \tilde{c}_1g_1 - \cdots - \tilde{c}_mg_m$. Then $vh \notin \{vg_1, \dots, vg_m\}$ and

$$f = (d_1 + c_n\tilde{c}_1)g_1 + \cdots + (d_m + c_n\tilde{c}_m)g_m + c_nh. \quad \square$$

Lemma 2.11. *Let M be an H -field extension of K and let $S \subseteq \Gamma_M$. Let $f_1, \dots, f_n \in K$ and suppose that f_1, \dots, f_n are C -linearly independent over $\{g \in K : vg > S\}$. Then f_1, \dots, f_n are C_M -linearly independent over $\{h \in M : vh > S\}$.*

Proof. Let $I = \{g \in K : vg > S\}$, so I is a C -linear subspace of K and f_1, \dots, f_n are C -linearly independent over I . Let $c_1, \dots, c_n \in C_M$, not all zero, and let $f = c_1f_1 + \cdots + c_nf_n$. We need to show that $vf \in S^\downarrow$. Since $f \in K[C_M]$, we may use Lemma 2.10 to take $d_1, \dots, d_m \in C_M^\times$ and $g_1 \succ g_2 \succ \cdots \succ g_m \in K^\times$ with $f = d_1g_1 + \cdots + d_mg_m$. By [3, Lemma 4.6.16], K and C_M are linearly disjoint over C , so we conclude that some nontrivial C -linear combination of f_1, \dots, f_n are contained in the C -linear span of g_1, \dots, g_m . As f_1, \dots, f_n are C -linearly independent over I , we conclude that some $g_i \notin I$. As I is \preceq -downward closed, we get that $g_1 \notin I$, so $vf = vg_1 \in S^\downarrow$. \square

3. H -FIELDS WITH CONSTANT POWER MAPS

In this section, we introduce the main object of study, H_{pow} -fields, and we prove our main technical result that allows us to “upgrade” various H -field extensions to H_{pow} -field extensions.

Let K be an H -field with constant field C . A **constant power map** on K is a binary function

$$(f, c) \mapsto f^c: K^> \times C \rightarrow K^>$$

that satisfies the following axioms (where f, g range over $K^>$ and c, d range over C):

(C1) $f^c f^d = f^{c+d}$, $(f^c)^d = f^{cd}$, and $f^1 = f$.

(C2) $(fg)^c = f^c g^c$.

(C3) If $f \sim 1$, then $f^c \sim 1$.

(C4) $(f^c)^\dagger = cf^\dagger$.

(C5) The map $c \mapsto 2^c$ is an ordered group isomorphism $C \rightarrow C^>$.

We refer to the map $c \mapsto 2^c: C \rightarrow C^>$ in (C5) as the **induced exponential on C** , and we consider C as an ordered exponential field with this induced exponential.

An H_{pow} -**field** is a real closed H -field K equipped with a constant power map. If K is an H_{pow} -field, then K is closed under powers, so Γ admits the structure of a C -vector space by Fact 2.1. If L is also an H_{pow} -field, then a map $\iota: K \rightarrow L$ is said to be an H_{pow} -**field embedding** if ι is an H -field embedding and $\iota(f^c) = \iota(f)^{\iota(c)}$ for all $f \in K^>$ and all $c \in C$. The notions of an H_{pow} -field extension and an H_{pow} -subfield are defined analogously.

Lemma 3.1. *Let K be an H_{pow} -field and let $E \supseteq C$ be a real closed H -subfield of K . If E is closed under powers, then E is an H_{pow} -subfield of K .*

Proof. Let $f \in E^>$ and $c \in C$. As E is closed under powers, we can find $g \in E^\times$ with $g^\dagger = cf^\dagger = (f^c)^\dagger$, so $g = df^c$ for some $d \in C^\times$. Since $C \subseteq E$, we get $f^c \in E$. \square

Lemma 3.2. *Let K be an H_{pow} -field and let \exp be an exponential on C with functional inverse \log . The following are equivalent:*

- (1) \exp coincides with the induced exponential on C .
- (2) $\exp(1) = 2$ and for all $c, d \in C$ with $d > 0$, we have $d^c = \exp(c \log d)$.

Proof. If \exp coincides with the induced exponential on C , then (C1) gives

$$\exp(1) = 2^1 = 2, \quad \exp(c \log d) = 2^{c \log d} = (2^{\log d})^c = d^c \quad \text{for } c, d \in C \text{ with } d > 0.$$

Conversely, if the conditions in (2) hold, then for $c \in C$, we have $2^c = \exp(c \log 2) = \exp(c)$, so \exp coincides with the induced exponential in C . \square

In many natural examples of H_{pow} -fields, such as \mathbb{T}_{pow} and maximal Hardy fields, the constant power map arises from an exponential function. Instead of verifying the constant power map axioms (C1)–(C5) hold directly, we show here that any exponential function on an H -field that is compatible with the derivation gives rise to a constant power map.

Proposition 3.3. *Let K be an H -field, let $\exp: K \rightarrow K^>$ be an exponential on K , and let \log be the compositional inverse of \exp . Suppose that \exp satisfies the identity $(\exp a)^\dagger = a'$ for $a \in K$. Then the map $(f, c) \mapsto \exp(c \log f): K^> \times C \rightarrow K^>$ is a constant power map on K .*

Proof. A routine computation shows that $(f, c) \mapsto \exp(c \log f)$ satisfies (C1) and (C2). The identity $\exp(a)^\dagger = a'$ also ensures that this map satisfies (C4), and it tells us that \exp restricts to an ordered group isomorphism $C \rightarrow C^>$. Using also that multiplication by $\log 2$ is an ordered group isomorphism $C \rightarrow C$, we get that $c \mapsto \exp(c \log 2): C \rightarrow C^>$ is an ordered group isomorphism, so (C5) holds. For (C3), let $f \in K$ with $f \sim 1$ and let $c \in C$. If $c = 0$, then $\exp(c \log f) = 1$. Suppose $c > 0$, and let $\varepsilon \in C$ with $0 < \varepsilon < 1$. We need to show that $1 - \varepsilon < \exp(c \log f) < 1 + \varepsilon$. Put

$$d_1 := \exp(c^{-1} \log(1 - \varepsilon)) \quad d_2 := \exp(c^{-1} \log(1 + \varepsilon))$$

so $d_1, d_2 \in C$ and $d_1 < 1 < d_2$. As $f \sim 1$, we have $d_1 < f < d_2$, so

$$1 - \varepsilon = \exp(c \log d_1) < \exp(c \log f) < \exp(c \log d_2) = 1 + \varepsilon.$$

The case that $c < 0$ is similar. \square

3.1. H_{pow} -field extensions from H -field extensions. In this subsection, let K be an H_{pow} -field. In axiomatizing the model companion of the theory of H -fields, Aschenbrenner, van den Dries, and van der Hoeven establish a great number of extension and embedding results for H -fields. To avoid reproving these results from scratch for H_{pow} -fields, we need a general criterion that tells us when we can “upgrade” these extensions and embeddings to also be compatible with a constant power map. We give this criterion in the following technical proposition, and we list various simplifications for the remainder of this subsection.

Proposition 3.4. *Let M be an H -field extension of K with real closed constant field C_M , and let \exp_M be an exponential on C_M extending the induced exponential on C . Let $(a_i)_{i \in I}$ be a tuple of elements in $M^>$ and suppose that*

- (i) *The tuple $(va_i)_{i \in I}$ spans $\mathbb{Q}\Gamma_M$ as a \mathbb{Q} -vector space over Γ .*
- (ii) *The tuple $(a_i^\dagger)_{i \in I}$ is C_M -linearly independent over the C_M -subspace of M generated by $(K^\times)^\dagger$ and $\{h \in M : vh > \Psi_M\}$.*

Then we have the following:

- (1) *M has a unique power closure M_{pow} up to M -isomorphism;*

- (2) There is a constant power map on M_{pow} with induced exponential \exp_M that makes M_{pow} into an H_{pow} -field extension of K ;
- (3) This constant power map is unique up to unique M -isomorphism: if L is an H_{pow} -field extension of K and $\iota: M \rightarrow L$ is an H -field embedding over K that restricts to an ordered exponential field embedding $C_M \rightarrow C_L$, then ι extends to a unique H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$.

Proof. First, we show that M has a unique power closure up to M -isomorphism, using Fact 2.5. Let $c_1, \dots, c_n \in C_M$ and $f_1, \dots, f_n \in M^\times$, and suppose towards contradiction that

$$\Psi_M < v(c_1 f_1^\dagger + \dots + c_n f_n^\dagger) < (\Gamma_M^>)'.$$

We assume n is minimal, so c_1, \dots, c_n are \mathbb{Q} -linearly independent by Lemma 2.9. By our assumption (i), we can find for each $j = 1, \dots, n$ a finitely supported tuple of rational numbers $(q_{i,j})_{i \in I}$ and an element $g_j \in K$ such that

$$f_j \asymp g_j \prod_{i \in I} a_i^{q_{i,j}},$$

where the product on the right belongs to M^{rc} , the real closure of M , so that the rational powers of the a_i make sense. Take also for each j an element $u_j \in M^{\text{rc}}$ with $u_j \asymp 1$ such that

$$f_j = g_j u_j \prod_{i \in I} a_i^{q_{i,j}}.$$

Then u_j^\dagger belongs to M , as it can be written as the sum of elements in M . For each $i \in I$, let $d_i := c_1 q_{i,1} + \dots + c_n q_{i,n}$. Then the tuple $(d_i)_{i \in I}$ is also finitely supported, and we have

$$c_1 f_1^\dagger + \dots + c_n f_n^\dagger = (c_1 g_1^\dagger + \dots + c_n g_n^\dagger) + (c_1 u_1^\dagger + \dots + c_n u_n^\dagger) + \sum_{i \in I} d_i a_i^\dagger.$$

Since $g_1^\dagger, \dots, g_n^\dagger \in (K^\times)^\dagger$ and since $u_1^\dagger, \dots, u_n^\dagger$ and $c_1 f_1^\dagger + \dots + c_n f_n^\dagger$ belong to the set $\{h \in M : vh > \Psi_M\}$, we may use our assumption (ii) to deduce that $d_i = 0$ for all i . As c_1, \dots, c_n are \mathbb{Q} -linearly independent, we must have that $q_{i,j} = 0$ for all $i \in I$ and all $j = 1, \dots, n$. Therefore, $f_j = g_j u_j$ for all j . Since $v(u_j^\dagger) \in (\Gamma_M^>)'$ and

$$v(c_1 f_1^\dagger + \dots + c_n f_n^\dagger) = v(c_1 g_1^\dagger + \dots + c_n g_n^\dagger + c_1 u_1^\dagger + \dots + c_n u_n^\dagger) < (\Gamma_M^>)',$$

we have $c_1 f_1^\dagger + \dots + c_n f_n^\dagger \asymp c_1 g_1^\dagger + \dots + c_n g_n^\dagger$. By replacing f_j by g_j for each j , we may arrange that each f_j is in K^\times . Now, since $v(c_1 f_1^\dagger + \dots + c_n f_n^\dagger) > \Psi_M$, we apply Lemma 2.11 with f_i^\dagger in place of f_i and Ψ_M in place of S . This gives $\tilde{c}_1, \dots, \tilde{c}_n \in C$, not all zero, with $v(\tilde{c}_1 f_1^\dagger + \dots + \tilde{c}_n f_n^\dagger) > \Psi$. We may as well assume that $\tilde{c}_n \neq 0$ and, dividing through by \tilde{c}_n , we may even assume that $\tilde{c}_n = 1$. Since K is an H_{pow} -field, we find $h \in K$ with $h^\dagger = \tilde{c}_1 f_1^\dagger + \dots + \tilde{c}_{n-1} f_{n-1}^\dagger + f_n^\dagger$. Then $v(h^\dagger) \in (\Gamma^>)'$, since $v(h^\dagger) > \Psi$. We have

$$c_1 f_1^\dagger + \dots + c_n f_n^\dagger = (c_1 - c_n \tilde{c}_1) f_1^\dagger + \dots + (c_{n-1} - c_n \tilde{c}_{n-1}) f_{n-1}^\dagger + c_n h^\dagger$$

Since $v(h^\dagger) \in (\Gamma^>)'$ and $v(c_1 f_1^\dagger + \dots + c_n f_n^\dagger) < (\Gamma_M^>)'$, we have

$$c_1 f_1^\dagger + \dots + c_n f_n^\dagger \asymp (c_1 - c_n \tilde{c}_1) f_1^\dagger + \dots + (c_{n-1} - c_n \tilde{c}_{n-1}) f_{n-1}^\dagger.$$

This contradicts the minimality of n .

We have now established that M has a unique power closure M_{pow} up to M -isomorphism. Now, we turn to constructing a constant power map on M_{pow} . For each $i \in I$, we put $\alpha_i := v a_i$. We also fix a C -linear basis $(\beta_j)_{j \in J}$ for Γ , and we choose for each j an element $b_j \in K^>$ with $v b_j = \beta_j$. We claim that $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$ together form a C_M -linear basis for $\Gamma_{M_{\text{pow}}} = C_M \Gamma_M$. Our assumption that $(\alpha_i)_{i \in I}$ spans $\mathbb{Q} \Gamma_M$ as a \mathbb{Q} -vector space over Γ ensures that $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$

span $\Gamma_{M_{\text{pow}}}$ as a C_M -vector space. For C_M -linear independence, let $(c_i)_{i \in I}$ and $(d_j)_{j \in J}$ be finitely supported tuples from C_M and suppose that

$$\sum_{i \in I} c_i \alpha_i + \sum_{j \in J} d_j \beta_j = 0.$$

By Lemma 2.2 (with M_{pow} in place of K), we get that $v(\sum_{i \in I} c_i a_i^\dagger + \sum_{j \in J} d_j b_j^\dagger) > \Psi_{M_{\text{pow}}}$. As Ψ_M is cofinal in $\Psi_{M_{\text{pow}}}$, we may use our assumption (ii) to see that each $c_i = 0$, so $v(\sum_{j \in J} d_j b_j^\dagger) > \Psi_{M_{\text{pow}}}$. Applying Lemma 2.2, this time to K and the tuple $(b_j)_{j \in J}$, we see that $(b_j^\dagger)_{j \in J}$ is C -linearly independent over $\{g \in K : vg > \Psi\}$. Applying Lemma 2.11 with the b_j^\dagger in place of the f_i and with $\Psi_{M_{\text{pow}}}$ in place of S , we conclude that each $d_j = 0$ as well.

With this claim established, we will define our constant power map, starting with the powers of the a_i and b_j . Fix a \mathbb{Q} -linear basis $(\hat{c}_\eta)_{\eta < \mu}$ for C with $\hat{c}_0 = 1$, and extend this to a \mathbb{Q} -linear basis $(\hat{c}_\eta)_{\eta < \nu}$ for C_M (so $\nu \geq \mu$, with equality if and only if $C_M = C$). For each $i \in I$ and each $0 < \eta < \nu$, we fix an element $a_{i,\eta} \in M_{\text{pow}}^>$ with $a_{i,\eta}^\dagger = \hat{c}_\eta a_i^\dagger$. We extend this to all $\eta < \nu$ by putting $a_{i,0} := a_i$. For each $j \in J$ and each $\mu \leq \eta < \nu$, we fix an element $b_{j,\eta} \in M_{\text{pow}}^>$ with $b_{j,\eta}^\dagger = \hat{c}_\eta b_j^\dagger$. We extend this to all $\eta < \nu$ by putting $b_{j,\eta} := b_j^{\hat{c}_\eta}$ for $\eta < \mu$. Now, let $c \in C_M$ and write $c = \sum_{\eta < \nu} q_\eta \hat{c}_\eta$ where $(q_\eta)_{\eta < \nu}$ is a finitely supported tuple from \mathbb{Q} . For $i \in I$, we put $a_i^c := \prod_{\eta < \nu} a_{i,\eta}^{q_\eta}$. Likewise, for $j \in J$, we put $b_j^c := \prod_{\eta < \nu} b_{j,\eta}^{q_\eta}$. Clearly, for each a_i we have

$$a_i^{\hat{c}_\eta} = a_{i,\eta} \text{ for all } \eta, \quad (a_i^c)^\dagger = c a_i^\dagger, \quad v(a_i^c) = c \alpha_i,$$

and likewise for the b_j . Now, we extend this constant power map to all of M_{pow} . Let $f \in M_{\text{pow}}^>$ and take finitely supported tuples $(c_i)_{i \in I}$ and $(d_j)_{j \in J}$ from C_M with $vf = \sum_{i \in I} c_i \alpha_i + \sum_{j \in J} d_j \beta_j$. Then

$$f = d(1 + \varepsilon) \prod_{i \in I} a_i^{c_i} \prod_{j \in J} b_j^{d_j}$$

for some unique $d \in C_M^>$ and $\varepsilon \in M_{\text{pow}}$ with $\varepsilon \prec 1$. We put $d^c := \exp_M(c \log_M d)$ and, using Lemma 2.3, we let $(1 + \varepsilon)^c$ be the unique element of $1 + \mathcal{O}_{M_{\text{pow}}}$ with $((1 + \varepsilon)^c)^\dagger = c(1 + \varepsilon)^\dagger$. We put

$$f^c := d^c (1 + \varepsilon)^c \prod_{i \in I} a_i^{c_i c} \prod_{j \in J} b_j^{d_j c}.$$

It is routine to check that this is indeed a constant power map. By Lemma 3.2, the induced exponential on C_M coincides with \exp_M (the identity $\exp_M(1) = 2$ follows from our assumption that \exp_M extends the induced exponential on C).

It remains to show that the claimed universal property holds. Let L be an H_{pow} -field extension of K , let $\iota: M \rightarrow L$ be an H -field embedding over K , and suppose that $\iota|_{C_M}: C_M \rightarrow C_L$ is an ordered exponential field embedding. We note that $(va_{i,\eta})_{i \in I, 0 < \eta < \nu}$ and $(vb_{j,\eta})_{j \in J, \mu \leq \eta < \nu}$ form a \mathbb{Q} -linear basis for $\Gamma_{M_{\text{pow}}}$ over $\mathbb{Q}\Gamma_M$. Let M_0 be the field extension of M generated by $(a_{i,\eta})_{i \in I, 0 < \eta < \nu}$ and $(b_{j,\eta})_{j \in J, \mu \leq \eta < \nu}$. As $a_{i,\eta}^\dagger = \hat{c}_\eta a_i^\dagger \in M$ and $b_{j,\eta}^\dagger = \hat{c}_\eta b_j^\dagger \in M$ for all $i \in I$, $j \in J$, and $\eta < \nu$, we may iteratively apply Lemma 1.1 to find an H -field embedding $M_0 \rightarrow L$ that sends $a_{i,\eta}$ to $\iota(a_i)^{\iota(\hat{c}_\eta)}$ and $b_{j,\eta}$ to $\iota(b_j)^{\iota(\hat{c}_\eta)}$ for each i, j, η . By Lemma 2.7, M_{pow} is the unique power closure of M_0 . As $\mathbb{Q}\Gamma_{M_0} = \Gamma_{M_{\text{pow}}}$, our embedding $M_0 \rightarrow L$ extends uniquely to an H -field embedding $j: M_{\text{pow}} \rightarrow L$ by Lemma 2.8. We claim that j is an H_{pow} -field embedding. Let $c = \sum_{\eta < \nu} q_\eta \hat{c}_\eta \in C_M$. For $i \in I$, we have

$$j(a_i^c) = \prod_{\eta < \nu} j(a_{i,\eta})^{q_\eta} = \prod_{\eta < \nu} \iota(a_i)^{q_\eta \iota(\hat{c}_\eta)} = \iota(a_i)^{\iota(c)} = j(a_i)^{j(c)}.$$

Likewise, $j(b_j^c) = j(b_j)^{j(c)}$. Let $\varepsilon \in M_{\text{pow}}$ with $\varepsilon \prec 1$. We have

$$j((1 + \varepsilon)^c)^\dagger = j(c(1 + \varepsilon)^\dagger) = j(c)j(1 + \varepsilon)^\dagger,$$

so Lemma 2.3 gives that $j((1 + \varepsilon)^c) = j(1 + \varepsilon)^{j(c)}$. Using Lemma 3.2 and our assumption that ι commutes with the induced exponentials on C_M and C_L , we get that $j(d^c) = j(d)^{j(c)}$ for all $d \in C_M^>$. As shown above, we may decompose any $f \in M_{\text{pow}}^>$ as a product

$$f = d(1 + \varepsilon) \prod_{i \in I} a_i^{c_i} \prod_{j \in J} b_j^{d_j}$$

where $d \in C_M^>$, $\varepsilon \prec 1$, and $(c_i)_{i \in I}$ and $(d_j)_{j \in J}$ are from C_M . From this it follows $j(f^c) = j(f)^{j(c)}$, so j is indeed an H_{pow} -field embedding. As for the uniqueness of j , we note that *any* H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$ extending ι must send $a_{i,\eta} = a_i^{\hat{c}_\eta}$ to $\iota(a_i)^{\iota(\hat{c}_\eta)}$ for each $i \in I$ and $\eta < \nu$; likewise with the $b_{j,\eta}$. The restriction of j to M_0 is uniquely determined by this requirement, so j is unique as well by Lemma 2.8. \square

In many applications of the above proposition, the extension M has the same constant field as K . In this case, we can use that $(K^\times)^\dagger$ and $\{h \in M : vh > \Psi_M\}$ are both C -subspaces of M and that \exp_M is necessarily the induced exponential on C to simplify the statement of the proposition:

Corollary 3.5. *Let M be an H -field extension of K with $C_M = C$, let $(a_i)_{i \in I}$ be a tuple of elements in $M^>$, and suppose that*

- (i) *The tuple $(va_i)_{i \in I}$ spans $\mathbb{Q}\Gamma_M$ as a \mathbb{Q} -vector space over Γ .*
- (ii) *The tuple $(a_i^\dagger)_{i \in I}$ is C -linearly independent over $(K^\times)^\dagger + \{h \in M : vh > \Psi_M\} \subseteq M$.*

Then M has a unique power closure M_{pow} , and there is a unique constant power map on M_{pow} that makes it an H_{pow} -field extension of K . If L is an H_{pow} -field extension of K , then any H -field embedding $M \rightarrow L$ over K extends uniquely to an H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$.

The statement of the proposition simplifies even further when the tuple $(va_i^\dagger)_{i \in I}$ is distinct. This situation occurs a few times in the remainder of the paper, so we record it here:

Corollary 3.6. *Let M be an H -field extension of K with $C_M = C$. Let $(\alpha_i)_{i \in I}$ be a tuple of nonzero elements in Γ_M and suppose that $(\alpha_i)_{i \in I}$ spans $\mathbb{Q}\Gamma_M$ as a \mathbb{Q} -vector space over Γ and that $(\alpha_i^\dagger)_{i \in I}$ is a tuple of distinct elements of $\Gamma_M \setminus \Psi$. Then M has a unique power closure M_{pow} , and there is a unique constant power map on M_{pow} that makes it an H_{pow} -field extension of K . If L is an H_{pow} -field extension of K , then any H -field embedding $M \rightarrow L$ over K extends uniquely to an H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$.*

Proof. Take for each $i \in I$ an element $a_i \in M^>$ with $va_i = \alpha_i$. By Corollary 3.5, we need only show that $(a_i^\dagger)_{i \in I}$ is C -linearly independent over

$$(K^\times)^\dagger + \{h \in M : vh > \Psi_M\}.$$

To see this, let $(c_i)_{i \in I}$ be a finitely supported tuple from C , not all zero, and let $g \in K^\times$. As the α_i^\dagger are distinct from one another and from vg^\dagger , we have

$$v(g^\dagger + \sum_{i \in I} c_i a_i^\dagger) = \min(\{\alpha_i^\dagger : c_i \neq 0\} \cup \{vg^\dagger\}) \in \Psi_M. \quad \square$$

Another frequently occurring situation is the case that M is an immediate extension of K .

Corollary 3.7. *Let M be an immediate H -field extension of K . Then M has a unique power closure M_{pow} . This power closure is an immediate extension of K , and there is a unique constant power map on M_{pow} that makes it an H_{pow} -field extension of K . If L is an H_{pow} -field extension*

of K , then any H -field embedding $M \rightarrow L$ over K extends uniquely to an H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$.

Proof. Apply Corollary 3.5, taking $(a_i)_{i \in I}$ to be the empty tuple. Fact 2.6 gives $\Gamma_{M_{\text{pow}}} = C\Gamma_M = C\Gamma = \Gamma$ so M_{pow} is an immediate extension of K . \square

The final situation we record here is the case where the value group of M is generated by one element over the value group of K .

Corollary 3.8. *Let M be an H -field extension of K with $C_M = C$ and $\Gamma_M = \Gamma \oplus \mathbb{Z}\alpha$ for some $\alpha \in \Gamma_M^>$. Then M has a unique power closure M_{pow} , and there is a unique constant power map on M_{pow} that makes it an H_{pow} -field extension of K . If L is an H_{pow} -field extension of K , then any H -field embedding $M \rightarrow L$ over K extends uniquely to an H_{pow} -field embedding $M_{\text{pow}} \rightarrow L$.*

Proof. Fix $a \in M^>$ with $va = \alpha$. By Corollary 3.5, it is enough to show that $a^\dagger \notin (K^\times)^\dagger + \{h \in M : vh > \Psi_M\}$. Let $g \in (K^\times)^\dagger$. If $v(a^\dagger + g^\dagger) = v(ag)^\dagger > \Psi_M$, then $ag \asymp 1$, so $a \asymp g^{-1}$, contradicting that $va \notin \Gamma$. \square

4. EXTENSIONS AND EMBEDDINGS OF H_{pow} -FIELDS

In this section, we collect the embedding results we need to prove our main theorem. We do this by combining various H -field embedding results from [3] with the tools established in the previous section on extending H -field embeddings to H_{pow} -field embeddings. For the remainder of this section, K is an H_{pow} -field.

4.1. Differentially algebraic H_{pow} -field extensions. In this subsection, we handle the various differentially algebraic extensions of our H_{pow} -field K . We begin with the newtonization and the Newton-Liouville closure.

Proposition 4.1. *Suppose that K is ω -free. Then there is a unique constant power map on K^{nt} , the newtonization of K , that makes it an immediate H_{pow} -field extension of K . If L is a newtonian H_{pow} -field extension of K , then there is an H_{pow} -field embedding $K^{\text{nt}} \rightarrow L$ over K .*

Proof. As K^{nt} is an immediate H -field extension of K , we may use Corollary 3.7 along with the universal property of the newtonization to see that the lemma holds for $(K^{\text{nt}})_{\text{pow}}$. Since K^{nt} is asymptotically d -algebraically maximal by Fact 1.9 and $(K^{\text{nt}})_{\text{pow}}$ is an immediate d -algebraic extension of K^{nt} by Corollary 3.7, we have $K^{\text{nt}} = (K^{\text{nt}})_{\text{pow}}$. \square

Lemma 4.2. *Let $s \in K$ with $s < 0$, and suppose that $v(s - a^\dagger) \in \Psi^\downarrow$ for all $a \in K^\times$. Then K has an H_{pow} -field extension $K(f)_{\text{pow}}$ with $f > 0$, $f^\dagger = s$, and $C_{K(f)_{\text{pow}}} = C$. This extension is unique up to K -isomorphism: given an H_{pow} -field extension L of K and $g \in L^>$ with $g^\dagger = s$, there is a unique H_{pow} -field embedding $K(f)_{\text{pow}} \rightarrow L$ that sends f to g .*

Proof. Lemma 10.5.20 in [3] gives an H -field extension $K(f)$ of K with

$$f > 0, \quad f^\dagger = s, \quad \Gamma_{K(f)} = \Gamma \oplus \mathbb{Z}vf, \quad C_{K(f)} = C,$$

and the desired universal property. The lemma follows from Corollary 3.8. \square

Proposition 4.3. *Suppose that K is ω -free. Then K has a newtonian Liouville closed H_{pow} -field extension K^{nl} that embeds over K into any newtonian Liouville closed H_{pow} -field extension of K . This extension K^{nl} is d -algebraic over K , with $C_{K^{\text{nl}}} = C$.*

Proof. Define a *Newton-Liouville tower* on K to be a tower $(K_\mu)_{\mu \leq \nu}$ of d -algebraic H_{pow} -field extensions of K , where

- (i) $K_0 = K$ and $K_\lambda = \bigcup_{\mu < \lambda} K_\mu$ whenever $0 < \lambda \leq \nu$ is a limit ordinal;

- (ii) if $\mu < \nu$ and K_μ is not newtonian, then $K_{\mu+1} = K_\mu^{\text{nt}}$ with the constant power map given in Proposition 4.1;
- (iii) if $\mu < \nu$, K_μ is newtonian, and there is $s \in K_\mu^<$ with $v(s - a^\dagger) \in \Psi^\downarrow$ for all $a \in K_\mu^\times$, then $K_{\mu+1} = K_\mu(f)_{\text{pow}}$ as described in Lemma 4.2, so $f > 0$ and $f^\dagger = s$.

If $(K_\mu)_{\mu \leq \nu}$ is a Newton-Liouville tower on K , then K_ν is a d-algebraic extension of K with the same field of constants as K , so maximal Newton-Liouville towers on K exist by Corollary 1.13 and Zorn's lemma.

Let $(K_\mu)_{\mu \leq \nu}$ be a maximal Newton-Liouville tower on K and let $K^{\text{nl}} := K_\nu$. By maximality, K^{nl} is newtonian, and we claim that it is Liouville closed as well. Lemma 1.6 tells us that K^{nl} is closed under integration. To see that K^{nl} is closed under exponential integration, let $s \in K^{\text{nl}}$ with $s < 0$. By maximality, there must be some nonzero $a \in K^{\text{nl}}$ with $v(s - a^\dagger) > \Psi$. Then Lemma 1.6 gives $f \in K^{\text{nl}}$ with $f^\dagger = s - a^\dagger$, so $(fa)^\dagger = s$.

Let L be a newtonian Liouville closed H_{pow} -field extension of K . The embedding properties in Proposition 4.1 and Lemma 4.2 give for each $\mu \leq \nu$ an H_{pow} -field embedding $\iota_\mu: K_\mu \rightarrow L$, where ι_0 is the inclusion $K \subseteq L$ and where ι_μ extends ι_η whenever $\eta < \mu \leq \nu$. \square

If K is an ω -free H_{pow} -field, then by Lemma 1.11, the underlying H -field of the H_{pow} -field extension K^{nl} constructed above is a Newton-Liouville closure of the H -field K , as defined in [3, Section 14.5]. That is, K^{nl} embeds over K into any newtonian Liouville closed H -field extension of K (not just the H_{pow} -field extensions). As any two Newton-Liouville closures of K are isomorphic over K by Fact 1.10, we see that *any* Newton-Liouville closure L of K admits a constant power map that makes L an H_{pow} -field extension of K .

Now, we turn to constant field extensions. Given an ordered field extension C^* of C , Propositions 10.5.15 and 10.5.16 in [3] give a unique ordering, valuation, and derivation on the field $K(C^*)$ such that $K(C^*)$ is an H -field extension of K with ordered constant field C^* , and we always consider $K(C^*)$ as an H -field in this way. With this valuation, $K(C^*)$ has the same value group as K .

Proposition 4.4. *Let C^* be a real closed ordered field extension of C and let \exp^* be an exponential on C^* extending the induced exponential on C . Then the H -field $K(C^*)$ has a unique power closure $K(C^*)_{\text{pow}}$, and there is a unique constant power map on $K(C^*)_{\text{pow}}$ extending the constant power map on K such that the induced exponential on C^* coincides with \exp^* . With this constant power map, $K(C^*)_{\text{pow}}$ has the following universal property: if L is an H_{pow} -field extension of K and $\iota: C^* \rightarrow C_L$ is an ordered exponential field embedding over C , then there is a unique H_{pow} -field embedding $K(C^*)_{\text{pow}} \rightarrow L$ over K extending ι .*

Proof. Apply Proposition 3.4 with $M = K(C^*)$, taking $(a_i)_{i \in I}$ to be the empty tuple. For the universal property, we first use the universal property of $K(C^*)$ given in [3, Proposition 10.5.16] to get an H -field embedding $K(C^*) \rightarrow L$ extending ι , and then we use the universal property in Proposition 3.4 to extend this to an H_{pow} -field embedding $K(C^*)_{\text{pow}} \rightarrow L$. \square

4.2. Constructing an ω -free H_{pow} -field extension. In this subsection, we will show that any H_{pow} -field can be extended to an ω -free H_{pow} -field.

Lemma 4.5. *Suppose that K has asymptotic integration. Then K has an H_{pow} -field extension with a gap and with constant field C .*

Proof. By [3, Corollary 11.4.10], K has a spherically complete immediate H -field extension L . Corollary 3.7 tells us that L_{pow} is an immediate H_{pow} -field extension of K , so $L = L_{\text{pow}}$ by spherical completeness. Using [3, Lemma 11.5.14] and the remarks following that lemma, we find $s \in L$ such that vf is a gap in $L(f)$ for any f in an H -field extension of L with $f^\dagger = s$ (in the terminology of [3], the element s creates a gap over L).

By [3, Lemma 11.5.10] we have $v(s - b^\dagger) \in \Psi_L^\downarrow$ for each $b \in L^\times$, so we may apply Lemma 4.2 to get an H_{pow} -field $L(f)_{\text{pow}}$ extending L with $f^\dagger = s$. Then vf is a gap in $L(f)_{\text{pow}}$ by the remarks above and Fact 2.5. \square

Lemma 4.6. *Let $s \in K$ and suppose that vs is a gap in K . Then K has a grounded H_{pow} -field extension $K(a)_{\text{pow}}$ with $a \prec 1$, $a' = s$, and $C_{K(a)_{\text{pow}}} = C$. This extension is unique up to K -isomorphism: given an H_{pow} -field extension L of K and $f \in \mathfrak{o}_L$ with $f' = s$, there is a unique H_{pow} -field embedding $K(a)_{\text{pow}} \rightarrow L$ that sends a to f .*

Proof. Using [3, Lemma 10.2.1 and Corollary 10.5.10], we find an H -field extension $K(a)$ of K with

$$a \prec 1, \quad a' = s, \quad \Gamma_{K(a)} = \Gamma \oplus \mathbb{Z}va, \quad C_{K(a)} = C,$$

and the desired universal property. The remark following [3, Lemma 10.2.1] tells us that $K(a)$ is grounded. The lemma follows from Corollary 3.8. \square

Using [3, Lemma 10.2.2 and Corollary 10.5.11] in place of Lemma 10.2.1 and Corollary 10.5.10 in the above lemma gives the following variant:

Lemma 4.7. *Let $s \in K$ and suppose that vs is a gap in K . Then K has a grounded H_{pow} -field extension $K(b)_{\text{pow}}$ with $b \succ 1$, $b' = s$, and $C_{K(b)_{\text{pow}}} = C$. This extension is unique up to K -isomorphism: Given an H_{pow} -field extension L of K and $g \in L$ with $g \succ 1$ and $g' = s$, there is a unique H_{pow} -field embedding $K(b)_{\text{pow}} \rightarrow L$ that sends b to g .*

Lemma 4.8. *Suppose that K is grounded. Then K has an ω -free H_{pow} -field extension $K_{\omega, \text{pow}}$ with constant field C that embeds over K into any Liouville closed H_{pow} -field extension of K .*

Proof. Lemma 11.7.17 in [3] gives an ω -free H -field extension K_ω of K with $C_{K_\omega} = C$ that embeds over K into any Liouville closed H -field extension of K . From the construction of this extension, we have

$$\Gamma_{K_\omega} = \Gamma \oplus \bigoplus_{n=1}^{\infty} \mathbb{Z}\beta_n$$

for some elements $\beta_1, \beta_2, \dots \in \Gamma_{K_\omega}$ with $\Psi < \beta_1^\dagger < \beta_2^\dagger < \dots$. We conclude by applying Corollary 3.6 with $M = K_\omega$. \square

The embedding property in the previous lemma can be strengthened: let L be an H_{pow} -field extension of K and suppose that for each $f \in L^\times$, there is $y \in L$ with $y' = f^\dagger$. Then $K_{\omega, \text{pow}}$ embeds over K into L (this uses that K_ω enjoys this stronger embedding property). However, we will only ever use the lemma in the case that L is Liouville closed.

Combining the previous lemmas, we obtain:

Corollary 4.9. *K has an ω -free H_{pow} -field extension with constant field C .*

4.3. Differentially transcendental H_{pow} -field extensions. In this section, we handle various H_{pow} -field extensions of K determined by cuts in K . We begin with adding a new “small infinite” element.

Lemma 4.10. *Let K be an ω -free H_{pow} -field. Then K has an H_{pow} -field extension $K\langle y \rangle_{\text{pow}}$ where $y > 0$ and $\Gamma^< < vy < 0$. This extension is unique up to K -isomorphism: given an H_{pow} -field extension L of K and $y^* \in L^>$ with $\Gamma^< < vy^* < 0$, there is a unique H_{pow} -field embedding $K\langle y \rangle_{\text{pow}} \rightarrow L$ that sends y to y^* . Additionally, the extension $K\langle y \rangle$ is grounded with $vy^\dagger = \max \Psi_{K\langle y \rangle_{\text{pow}}}$.*

Proof. Saturation gives us an H -field extension $K\langle y \rangle$ of K where $y > 0$ and $\Gamma^< < vy < 0$. By [3, Corollary 13.6.8], this extension is unique up to K -isomorphism: given an H -field extension L of

K and $y^* \in L^>$ with $\Gamma^< < vy^* < 0$, there is a unique H -field embedding $K\langle y \rangle \rightarrow L$ that sends y to y^* . Moreover, by [3, Lemma 13.4.4 and Proposition 13.6.7], we have

$$C_{K\langle y \rangle} = C, \quad \Gamma_{K\langle y \rangle} = \Gamma \oplus \mathbb{Z}vy \oplus \mathbb{Z}vy^\dagger, \quad \max \Psi_{K\langle y \rangle} = vy^\dagger.$$

In order to apply Corollary 3.5, we need only show that y^\dagger and $y^{\dagger\dagger}$ are C -linearly independent over

$$(K^\times)^\dagger + \{g \in K\langle y \rangle : vg > \Psi_{K\langle y \rangle}\}.$$

Let $c, d \in C$ and $f \in K^\times$ with

$$v(cy^\dagger + dy^{\dagger\dagger} + f^\dagger) > \Psi_{K\langle y \rangle}.$$

Since $vy^\dagger = \max \Psi_{K\langle y \rangle}$, this gives

$$v(cy^\dagger + dy^{\dagger\dagger} + f^\dagger) > vy^\dagger.$$

If $d \neq 0$, then we may arrange that $d = 1$. Applying [3, Corollary 9.8.6] with $\beta = vy^\dagger$, we get that $\psi(vy^\dagger + \gamma) \in \Psi$ for all $\gamma \in \Gamma$. Thus, we have

$$v(y^{\dagger\dagger} + f^\dagger) = v(y^\dagger f)^\dagger = \psi(vy^\dagger + vf) \in \Psi < vy^\dagger.$$

It follows that $v(cy^\dagger + dy^{\dagger\dagger} + f^\dagger) < vy^\dagger$, a contradiction. Thus, $d = 0$, and $v(cy^\dagger + f^\dagger) > vy^\dagger$. If $c \neq 0$, then $v(cy^\dagger) = vy^\dagger$, so $v(f^\dagger) = vy^\dagger$ as well, contradicting that $vy^\dagger \notin \Gamma$. Thus, c must also equal 0. \square

Now we establish an analog of the results in [3, Section 16.1] for H_{pow} -fields.

Lemma 4.11. *Let K be an ω -free newtonian Liouville closed H_{pow} -field, let $K\langle f \rangle$ be an H -field extension of K , and suppose that $C_{K\langle f \rangle} = C$ and that $K\langle y \rangle$ is not an immediate extension of K for all $y \in K\langle f \rangle$. Then $K\langle f \rangle$ has a unique power closure $K\langle f \rangle_{\text{pow}}$, and there is a constant power map on $K\langle f \rangle_{\text{pow}}$ that makes it an H_{pow} -field extension of K . If L is an H_{pow} -field extension of K and $g \in L$ realizes the same cut as f over K , then there is a unique H_{pow} -field embedding $K\langle f \rangle_{\text{pow}} \rightarrow L$ that sends f to g .*

Proof. Lemma 16.1.2 in [3] gives elements $(\beta_n)_{n \in \mathbb{N}}$ from $\Gamma_{K\langle f \rangle}$ such that

$$\Gamma_{K\langle f \rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}\beta_n, \quad \beta_0^\dagger < \beta_1^\dagger < \dots \text{ is an increasing tuple of elements in } \Gamma_{K\langle f \rangle} \setminus \Gamma.$$

We conclude by Corollary 3.6 with $M = K\langle f \rangle$. The universal property for $K\langle f \rangle_{\text{pow}}$ follows from the embedding property in that corollary, together with the universal property for $K\langle f \rangle$ given in [3, Proposition 16.1.5]. \square

For our proof of local ω -minimality, we need the following variant of the above lemma, where we drop some assumptions on K but now assume that our new element is infinite relative to K :

Lemma 4.12. *Let K be an ω -free Liouville closed H_{pow} -field. Then K has an ω -free H_{pow} -field extension $K\langle a \rangle_{\text{pow}}$ with $a > K$. This extension is unique up to K -isomorphism: for any element $b > K$ in an H_{pow} -field extension L of K , there is a unique H_{pow} -field embedding $K\langle a \rangle_{\text{pow}} \rightarrow L$ that sends a to b .*

Proof. Compactness gives a in an H -field extension of K with $a > K$. Put $\beta_0 = va$ and for $n \in \mathbb{N}$, put $\beta_{n+1} := \beta_n^\dagger$. Lemmas 16.6.9 and 16.6.10 in [3] tells us that the ordered differential field $K\langle a \rangle$ is an H -field extension of K with

$$C_{K\langle a \rangle} = C, \quad \Gamma_{K\langle a \rangle} = \Gamma \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\beta_n, \quad \beta_0 < \beta_1 < \beta_2 < \dots < \Gamma,$$

and the proofs of those lemmas give the desired embedding property for $K\langle a \rangle$. The lemma follows by applying Corollary 3.6 with $M = K\langle a \rangle$. \square

5. MODEL COMPLETENESS FOR \mathbb{T}_{pow}

In this section, we put together the previously established embedding lemmas to prove our main results.

Proposition 5.1. *Let K and L be ω -free newtonian Liouville closed H_{pow} -fields and assume that the underlying ordered set of L is $|K|^+$ -saturated and the cofinality of $\Gamma_L^<$ is greater than $|\Gamma|$. Let E be an ω -free H_{pow} -subfield of K with $C_E = C$ and let $\iota: E \rightarrow L$ be an H_{pow} -field embedding. Then ι extends to an H_{pow} -field embedding $K \rightarrow L$.*

Proof. We can assume $E \neq K$, in which case it suffices to show that ι can be extended to an embedding of an ω -free H_{pow} -subfield of K that properly contains E . If E is not also newtonian and Liouville closed, then we use Proposition 4.3 to identify E^{nl} with an H_{pow} -subfield of K and to extend ι to an H_{pow} -field embedding $E^{\text{nl}} \rightarrow L$. We assume for the remainder of the proof that E is an ω -free newtonian Liouville closed H_{pow} -subfield of K .

Suppose $\Gamma_E^<$ is not cofinal in $\Gamma^<$, so there is $y \in K^>$ with $\Gamma_E^< < \nu y < 0$. By our cofinality assumption on $\Gamma_L^<$ there is $y^* \in L^>$ with $\Gamma_{\iota(E)}^< < \nu y^* < 0$. Using Lemma 4.10, we extend ι to an H_{pow} -field embedding $E\langle y \rangle_{\text{pow}} \rightarrow L$. As $E\langle y \rangle_{\text{pow}}$ is grounded, we can use Lemma 4.8 to further extend ι to an embedding of an ω -free H_{pow} -subfield of K containing y . We assume for the remainder of the proof that $\Gamma_E^<$ is cofinal in $\Gamma^<$. Then every H -subfield of K containing E is ω -free by Fact 1.3.

Next, suppose $E\langle z \rangle$ is an immediate extension of E for some $z \in K \setminus E$. The saturation assumption on L gives $z^* \in L$ that realizes the ι -image of the cut over $\iota(E)$ that z realizes over E . By Fact 1.7 we may extend ι to an H -field embedding of $E\langle z \rangle$ into L that sends z to z^* . By Corollary 3.7, this extends to an H_{pow} -field embedding $E\langle z \rangle_{\text{pow}} \rightarrow L$.

Finally, assume E has no proper immediate extensions in K . Take $f \in K \setminus E$ and, using the saturation assumption on L , take $g \in L$ that realizes the ι -image of the cut over $\iota(E)$ that f realizes over E . By Lemma 4.11 we may extend ι to an H_{pow} -field embedding $K\langle f \rangle_{\text{pow}} \rightarrow L$. \square

Here is a useful consequence of Proposition 5.1.

Corollary 5.2. *Let K and L be as in the statement of Proposition 5.1. If both K and L have small derivation, then any ordered exponential field embedding $\iota: C \rightarrow C_L$ extends to an H_{pow} -field embedding $K \rightarrow L$. The same is true if both K and L have large derivation.*

Proof. We first consider the case that K and L have small derivation. As K is Liouville closed, there is $f \in K$ with $f' = 1$. Additionally, $f \succ 1$ since K has small derivation. Since L is also Liouville closed with small derivation, there is $g \in L$ with $g \succ 1$ and $g' = 1$. Applying Lemma 4.7 with $s = 1$, we find a grounded H_{pow} -field $E = C\langle b \rangle_{\text{pow}}$ with $b' = 1$ that embeds into both K and L . By Lemma 4.8, these embeddings extend to embeddings of an ω -free H_{pow} -field $E_{\omega, \text{pow}}$ into K and L . Identifying the images of $E_{\omega, \text{pow}}$ in K and L , we arrange that K and L share a common ω -free H_{pow} -subfield with the same field of constants as K . Then Proposition 5.1 gives an H_{pow} -field embedding $K \rightarrow L$.

Now suppose K and L have large derivation. Again, since K is Liouville closed, there is $y \in K$ with $y' = -1$. Then $y \preccurlyeq 1$ since K has large derivation, so by subtracting a constant from y , we may assume that $y \prec 1$. The same holds for L , so we may apply the same argument above, this time using Lemma 4.6 in place of Lemma 4.7. \square

5.1. Model completeness and completeness. To get a model completeness result, we need to remove the assumption that $C_E = C$ in Proposition 5.1. To do this, we impose an additional requirement on the constant fields of K , L , and E . An H_{pow} -field K is said to have **real exponential constant field** if its constant field C equipped with the induced exponential $c \mapsto 2^c$ is a model of $\text{Th}(\mathbb{R}, x \mapsto 2^x)$.

Corollary 5.3. *Let E, K, L and ι be as in the statement of Proposition 5.1, except we drop the assumption that $C_E = C$. Assume, in addition, that the underlying ordered set of C_L is $|C|^+$ -saturated and that E, K , and L all have real exponential constant fields. Then ι extends to an H_{pow} -field embedding $K \rightarrow L$.*

Proof. The theory of $(\mathbb{R}, x \mapsto 2^x)$ is model complete and o-minimal by Wilkie’s theorem [25]. The saturation assumption on the underlying ordered set of C_L gives us that C_L is saturated as an ordered exponential field by o-minimality. By model completeness, the ordered exponential field embedding $\iota|_{C_E}: C_E \rightarrow C_L$ extends to an ordered exponential field embedding $j: C \rightarrow C_L$. By Proposition 4.4 there is a unique H_{pow} -field embedding $E(C)_{\text{pow}} \rightarrow L$ that extends both ι and j . Since $E(C)_{\text{pow}}$ is d-algebraic over E , it is ω -free by Fact 1.3. Now apply Proposition 5.1 with $E(C)_{\text{pow}}$ in place of E . \square

Let $\mathcal{L}_{\text{pow}} := \{+, \times, 0, 1, \leq, \preceq, \partial, \rho\}$, where \leq and \preceq are binary relation symbols, ∂ is a unary function symbol, and ρ is a binary function symbol. We view each H_{pow} -field K as an \mathcal{L}_{pow} -structure in the obvious way, where ρ is defined to be identically zero off of $K^> \times C$ and $\rho(f, c) = f^c$ for $(f, c) \in K^> \times C$. Consider the following \mathcal{L}_{pow} -theories:

- (1) T_{pow} , whose models are H_{pow} -fields with real exponential constant field.
- (2) T_{pow}^ω , whose models are ω -free H_{pow} -fields with real exponential constant field.
- (3) $T_{\text{pow}}^{\text{nl}}$, whose models are ω -free newtonian Liouville closed H_{pow} -fields with real exponential constant field.

Theorem 5.4. *The \mathcal{L}_{pow} -theory $T_{\text{pow}}^{\text{nl}}$ is model complete. It is the model companion of T_{pow} and the model completion of T_{pow}^ω .*

Proof. The fact that $T_{\text{pow}}^{\text{nl}}$ is the model completion of T_{pow}^ω follows immediately from Proposition 4.3, Corollary 5.3, and a standard model completion criterion; see [16, page viii]. Corollary 4.9 gives that $T_{\text{pow}}^{\text{nl}}$ is the model companion of T_{pow} . \square

We can use Theorem 5.4 and Corollary 5.2 to characterize the completions of $T_{\text{pow}}^{\text{nl}}$. Let $T_{\text{pow,sm}}^{\text{nl}}$ be the \mathcal{L}_{pow} -theory extending $T_{\text{pow}}^{\text{nl}}$ whose models have small derivation and let $T_{\text{pow,lg}}^{\text{nl}}$ be the \mathcal{L}_{pow} -theory extending $T_{\text{pow}}^{\text{nl}}$ whose models have large derivation.

Theorem 5.5. *$T_{\text{pow,sm}}^{\text{nl}}$ and $T_{\text{pow,lg}}^{\text{nl}}$ are the two completions of $T_{\text{pow}}^{\text{nl}}$.*

Proof. Let $K, L \models T_{\text{pow,sm}}^{\text{nl}}$ and assume L is $|K|^+$ -saturated. Then C_L is $|C|^+$ -saturated, so there is an ordered exponential field embedding $\iota: C \rightarrow C_L$ since C and C_L are elementarily equivalent; see [3, Corollary B.9.5]. This, in turn, extends to an embedding $j: K \rightarrow L$ by Corollary 5.2. Then j is elementary since $T_{\text{pow}}^{\text{nl}}$ is model complete, so K and L are elementarily equivalent. This shows that $T_{\text{pow,sm}}^{\text{nl}}$ is complete, and the same proof shows that $T_{\text{pow,lg}}^{\text{nl}}$ is complete. \square

Our axiomatization of $T_{\text{pow}}^{\text{nl}}$ is effective, with the possible exception of the “real exponential constant field” axiom. Thus, we have the following:

Corollary 5.6. *The $T_{\text{pow}}^{\text{nl}}$ and its two completions $T_{\text{pow,sm}}^{\text{nl}}$ and $T_{\text{pow,lg}}^{\text{nl}}$ are decidable relative to the theory of $\text{Th}(\mathbb{R}, x \mapsto 2^x)$.*

Decidability for the theory of $(\mathbb{R}, x \mapsto 2^x)$ (which is, of course, equivalent to decidability for the theory of \mathbb{R}_{exp}) is closely connected to transcendental number theory and is implied by Schanuel’s conjecture; see [20].

Recall that \mathbb{T}_{pow} is the expansion of the ordered valued differential field of transseries with the constant power map $(f, r) \mapsto f^r = \exp(r \log f)$. The exponential on \mathbb{T} satisfies the identity $(\exp f)^\dagger = f'$, so \mathbb{T}_{pow} is an H_{pow} -field by Proposition 3.3. The restriction of the exponential on

\mathbb{T} to its constant field \mathbb{R} is the real exponential, so the induced exponential on \mathbb{R} coincides with $x \mapsto 2^x$. Thus, we have

Corollary 5.7. *The \mathcal{L}_{pow} -theory of \mathbb{T}_{pow} is model complete, and it is completely axiomatized by $T_{\text{pow,sm}}^{\text{nl}}$.*

By Lemma 3.1, any real closed H -subfield of \mathbb{T} that contains \mathbb{R} and is closed under powers is an H_{pow} -subfield of \mathbb{T}_{pow} . Thus, any Liouville closed ω -free newtonian subfield of \mathbb{T} containing \mathbb{R} is an elementary substructure of \mathbb{T}_{pow} . These substructures include the subfield \mathbb{T}^{da} of transseries that are d-algebraic over \mathbb{Q} , as well as the subfield \mathbb{T}_{g} of grid-based transseries; see [15] for more information about \mathbb{T}_{g} .

5.2. Automorphisms and non-definability of the exponential. In terms of definability, \mathbb{T}_{pow} expands the ordered valued differential field \mathbb{T} , and it is a reduct of the ordered valued differential exponential field \mathbb{T}_{exp} . Clearly, \mathbb{T}_{pow} properly expands \mathbb{T} , as the constant field in \mathbb{T} is stably embedded as a real closed ordered field [3, Proposition 16.6.7] and \mathbb{T}_{pow} defines the function $c \mapsto 2^c$ on the constants. Below, we extend this somewhat by showing that there are automorphisms of \mathbb{T} that fix the constants and don't preserve the constant powers map. Additionally, we demonstrate that \mathbb{T}_{pow} is a strict reduct of \mathbb{T}_{exp} by identifying an automorphism of \mathbb{T}_{pow} that does not preserve the exponential. To do this, we need the following (unpublished) result of Aschenbrenner, van den Dries, and van der Hoeven on strongly \mathbb{R} -linear automorphisms of \mathbb{T} (where an automorphism $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ is said to be *strongly \mathbb{R} -linear* if it fixes \mathbb{R} and commutes with infinite sums).

Fact 5.8. *Let $\alpha: \mathbb{T} \rightarrow \mathbb{R}$ be an additive map that vanishes on the convex hull of \mathbb{R} . Then there is a unique strongly \mathbb{R} -linear H -field automorphism σ_α of \mathbb{T} such that*

$$\sigma_\alpha(x) = x, \quad \sigma_\alpha(\exp f) = \exp(\sigma_\alpha(f) + \alpha(f)) \text{ for all } f \in \mathbb{T}.$$

Lemma 5.9. *Let α be as above. Then for $f \in \mathbb{T}^>$ and $r \in \mathbb{R}$, we have $\sigma_\alpha(f^r) = \sigma_\alpha(f)^r$ if and only if $\alpha(r \log f) = r\alpha(\log f)$. In particular, σ_α is an H_{pow} -field automorphism of \mathbb{T}_{pow} if and only if α is \mathbb{R} -linear.*

Proof. Let $f \in \mathbb{T}^>$, and let $r \in \mathbb{R}$. Writing $f = \exp(\log f)$, we get $\sigma_\alpha(f) = \exp(\sigma_\alpha(\log f) + \alpha(\log f))$. Thus, we have

$$\begin{aligned} \sigma_\alpha(f^r) &= \sigma_\alpha(\exp(r \log f)) = \exp(\sigma_\alpha(r \log f) + \alpha(r \log f)), \\ \sigma_\alpha(f)^r &= \exp(r \log \sigma_\alpha(f)) = \exp(r\sigma_\alpha(\log f) + r\alpha(\log f)). \end{aligned}$$

Since σ_α is \mathbb{R} -linear, we have $\sigma_\alpha(f^r) = \sigma_\alpha(f)^r$ if and only if $\alpha(r \log f) = r\alpha(\log f)$. \square

Corollary 5.10.

- (1) *Let $f \in \mathbb{T}^>$ be either infinite or infinitesimal and let $r \in \mathbb{R}$ be irrational. Then there is a strongly \mathbb{R} -linear H -field automorphism $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ such that $\sigma(f^r) \neq \sigma(f)^r$.*
- (2) *Let $g \in \mathbb{T}$ be infinite. Then there is a strongly \mathbb{R} -linear H_{pow} -field automorphism $\tau: \mathbb{T}_{\text{pow}} \rightarrow \mathbb{T}_{\text{pow}}$ such that $\tau(\exp g) \neq \exp \tau(g)$.*

Proof. For (1), note that $\log f$ is infinite, so we may take a \mathbb{Q} -vector space decomposition $\mathbb{T} = \mathbb{Q} \log(f) \oplus V$ where V contains the convex hull of \mathbb{R} in \mathbb{T} , as well as the element $r \log f$. Let $\alpha: \mathbb{T} \rightarrow \mathbb{Q}$ be the projection map given by this decomposition, so $\alpha(q \log f + v) = q$ for $q \in \mathbb{Q}$ and $v \in V$. Then $r\alpha(\log f) = r$, whereas $\alpha(r \log f) = 0$, so Lemma 5.9 gives $\sigma_\alpha(f^r) \neq \sigma_\alpha(f)^r$.

Now for (2), take an \mathbb{R} -vector space decomposition $\mathbb{T} = \mathbb{R}g \oplus W$, where W contains the convex hull of \mathbb{R} in \mathbb{T} . Let $\beta: \mathbb{T} \rightarrow \mathbb{R}$ be the projection map given by this decomposition, so $\beta(rg + w) = r$ for $r \in \mathbb{R}$ and $w \in W$. As β is \mathbb{R} -linear, Lemma 5.9 gives that σ_β is an H_{pow} -field automorphism. It remains to observe that

$$\sigma_\beta(\exp g) = \exp(\sigma_\beta(g) + 1) \neq \exp(\sigma_\beta(g)). \quad \square$$

Corollary 5.10 shows that no restriction of the constant power map to the positive infinite or infinitesimal elements of \mathbb{T} is definable in \mathbb{T} , nor is any restriction of the exponential to the infinite elements of \mathbb{T} definable in \mathbb{T}_{pow} . Of course, the restriction of \exp to \mathbb{R} is definable in \mathbb{T}_{pow} . Moreover, the restriction of \exp to the maximal ideal of \mathbb{T} is even definable in \mathbb{T} , as for $\varepsilon \in \mathbb{T}$ with $\varepsilon \prec 1$, the element $\exp(\varepsilon)$ is the unique solution to the differential equation $y' = \varepsilon'y$ with $y \sim 1$; see [16, Lemma 8.8]. Thus, the restriction of \exp to the convex hull of \mathbb{R} is definable in \mathbb{T}_{pow} .

5.3. Local \mathfrak{o} -minimality and induced structure on the constants. In this subsection, let $K \models T_{\text{pow}}^{\text{nl}}$, and fix a $|K|^+$ -saturated elementary extension K^* of K . Here, we describe the structure of unary definable subsets of K , as well as the induced structure on the constant field.

Corollary 5.11. *Let $X \subseteq K$ be $\mathcal{L}_{\text{pow}}(K)$ -definable.*

- (1) *There is $f \in K$ such that $(f, +\infty)$ is either contained in or disjoint from X .*
- (2) *There is $g \in K$ with $g > C$ such that $\{y \in K : C < y < g\}$ is either contained in or disjoint from X .*
- (3) *For each $b \in K$, there is $h \in K$ with $h > b$ such that (b, h) is either contained in or disjoint from X .*

Proof. Fix elements $a_1, a_2 \in K^*$. For (1), suppose that $a_1, a_2 > K$. By a standard model-theoretic argument, it suffices to show that a_1 and a_2 have the same \mathcal{L}_{pow} -type over K . Lemma 4.12 tells us that $K\langle a_i \rangle_{\text{pow}} \models T_{\text{pow}}^{\text{ow}}$ for $i = 1, 2$ and that there is an H_{pow} -field isomorphism $K\langle a_1 \rangle_{\text{pow}} \simeq K\langle a_2 \rangle_{\text{pow}}$ over K that sends a_1 to a_2 . As $T_{\text{pow}}^{\text{nl}}$ is the model completion of $T_{\text{pow}}^{\text{ow}}$ by Theorem 5.4, this isomorphism is a partial $\mathcal{L}_{\text{pow}}(K)$ -elementary map $K^* \rightarrow K^*$. Note that (3) follows from (1), using fractional linear transformations.

For (2), we now assume that $C < a_1, a_2 < \{y \in K : y > C\}$. We again need to show that a_1 and a_2 have the same \mathcal{L}_{pow} -type over K . This time, we use Lemma 4.10 to get an H_{pow} -field isomorphism $K\langle a_1 \rangle_{\text{pow}} \simeq K\langle a_2 \rangle_{\text{pow}}$ over K that sends a_1 to a_2 . Using Lemma 4.8, this extends to an H_{pow} -field isomorphism $K_1 \simeq K_2$ over K , where $K\langle a_i \rangle_{\text{pow}} \subseteq K_i \models T_{\text{pow}}^{\text{ow}}$ for $i = 1, 2$. Again, this isomorphism is a partial $\mathcal{L}_{\text{pow}}(K)$ -elementary map $K^* \rightarrow K^*$ by Theorem 5.4. \square

Both properties above were established in much the same way for T^{nl} in [3, Section 16.6].

Corollary 5.12. *The constant field C of K is stably embedded as a model of $\text{Th}(\mathbb{R}, x \mapsto 2^x)$. That is, any $\mathcal{L}_{\text{pow}}(K)$ -definable subset $X \subseteq C^n$ is definable in the exponential field $(C, x \mapsto 2^x)$.*

Proof. Let \mathcal{L}_{exp} be the language of ordered exponential fields, and fix tuples $c_1, c_2 \in C_{K^*}^n$ with the same \mathcal{L}_{exp} -type over C . By a standard model-theoretic argument, it suffices to show that c_1 and c_2 have the same \mathcal{L}_{pow} -type over K . For each i , let C_i be the \mathcal{L}_{exp} -definable closure of $C(c_i)$ in C_{K^*} , so C_i is an \mathcal{L}_{exp} -elementary extension of C and there is an ordered exponential field isomorphism $C_1 \rightarrow C_2$ over C that sends c_1 to c_2 . This extends by Proposition 4.4 to an H_{pow} -field isomorphism $K(C_1)_{\text{pow}} \rightarrow K(C_2)_{\text{pow}}$ over K . This isomorphism is a partial $\mathcal{L}_{\text{pow}}(K)$ -elementary map $K^* \rightarrow K^*$ by Theorem 5.4, since each $K(C_i)_{\text{pow}}$ is \mathfrak{o} -free by Fact 1.3. \square

In T^{nl} , the constant field is stably embedded as a real closed ordered field [3, Proposition 16.6.7]. That result was established using quantifier elimination, as opposed to our method above using constant field extensions.

5.4. Non-uniform constant powers. Let $K \models T_{\text{pow}}^{\text{nl}}$, and let $\mathcal{L}_{C\text{-pow}}$ extend the language of ordered rings by constant symbols for each element of the constant field C along with unary function symbols for raising to each of these powers. That is,

$$\mathcal{L}_{C\text{-pow}} := \{+, \times, 0, 1, \leq, (c)_{c \in C}, (x \mapsto x^c)_{c \in C}\}.$$

Let $\mathcal{L}_{C\text{-pow}}^* := \mathcal{L}_{C\text{-pow}} \cup \{\preceq, \partial\}$. We construe K as an $\mathcal{L}_{C\text{-pow}}^*$ -structure in the natural way. In doing this, we essentially forget the uniformity in our constant power map. Note that C is an $\mathcal{L}_{C\text{-pow}}$ -substructure of K .

Let T denote the complete $\mathcal{L}_{C\text{-pow}}$ -theory of C . As the exponential field C is a model of the o-minimal theory $\text{Th}(\mathbb{R}, x \mapsto 2^x)$, the theory of the reduct T is also o-minimal.

Proposition 5.13. *The $\mathcal{L}_{C\text{-pow}}^*$ -structure K is an H_T -field in the sense of [18]. That is, the reduct of K to the language $\mathcal{L}_{C\text{-pow}}$ is an elementary extension of C , and the derivation on K is a T -derivation in the sense of [13].*

Proof. Let us first prove this proposition when $K = \mathbb{T}_{\text{pow}}$. In this case, $C = \mathbb{R}$, and the theory T is just the theory of the real field with real powers, as studied by Miller [21]. Let T_{exp} be the elementary theory of \mathbb{R}_{exp} in the language of ordered exponential fields with a constant symbol for each real number. The underlying exponential field of \mathbb{T}_{exp} is a model of T_{exp} by [11, Corollary 2.8], and the derivation on \mathbb{T}_{exp} is a T_{exp} -derivation; see [13, Example 2.16]. As T is a reduct of T_{exp} , this shows that the $\mathcal{L}_{\mathbb{R}\text{-pow}}$ -reduct of \mathbb{T}_{pow} is a model of T and that the derivation on \mathbb{T}_{pow} is a $\mathcal{L}_{\mathbb{R}\text{-pow}}$ -derivation.

We now show that the proposition holds whenever $K \models T_{\text{pow,sm}}^{\text{nl}}$. In this case, K is elementarily equivalent to \mathbb{T}_{pow} . The statements that K is an $\mathcal{L}_{C\text{-pow}}$ -elementary extension of C and that the derivation on K is a T -derivation can be written as a collection of \mathcal{L}_{pow} -formulas (first, rewrite them as $\mathcal{L}_{\text{pow}}(C)$ -formulas in the obvious way, then assert that they hold for all tuples of constants). Thus, the proposition holds for K since it holds for \mathbb{T}_{pow} .

Finally, if $K \models T_{\text{pow,lg}}^{\text{nl}}$, then by replacing the derivation ∂ by a compositional conjugate $\phi^{-1}\partial$ for some suitably chosen $\phi \in K^>$, we obtain a model $K^\phi \models T_{\text{pow,sm}}^{\text{nl}}$. Then the proposition holds for K^ϕ , so it holds for K as well, as being a T -derivation is invariant under compositional conjugation; see [17, Section 2.2]. \square

6. HARDY FIELDS AND SURREAL NUMBERS

In this section, we show that the results established on Hardy fields and surreal numbers in [4, 5] continue to hold when these structures are expanded by their natural constant power maps.

6.1. Hardy fields and transfer theorems. Recall that any Hardy field containing \mathbb{R} is an H -field with small derivation and constant field \mathbb{R} .

Fact 6.1 ([7, Theorems 5.1 and 5.3]). *Any maximal Hardy field is Liouville closed: it is real closed, contains \mathbb{R} , and is closed under taking integrals, exponentials, and logarithms of positive elements.*

Fact 6.2 ([5]). *Any maximal Hardy field is ω -free and newtonian.*

By Fact 6.1, any maximal Hardy field M is an H -field with an exponential exp that sends the germ f to the germ $\text{exp } f$. This exponential extends the real exponential on \mathbb{R} , and it satisfies the identity $(\text{exp } f)^\dagger = f'$. Combining this with Proposition 3.3, Fact 6.2, and Lemma 3.1, we obtain the following.

Corollary 6.3. *Any maximal Hardy field, equipped with the constant power map $(f, r) \mapsto f^r = \text{exp}(r \log f)$, is a model of $T_{\text{pow,sm}}^{\text{nl}}$. Let $H \supseteq \mathbb{R}$ be a real closed Hardy field and suppose that H is closed under powers (as an H -field). Then H is closed under this constant power map, and with this map, H is a model of T_{pow} .*

Theorem 6.4. *Let $H \supseteq \mathbb{R}$ be a real closed Hardy field that is closed under powers, and let $\iota: H \rightarrow \mathbb{T}_{\text{pow}}$ be an H_{pow} -field embedding. Let σ be an $\mathcal{L}_{\text{pow}}(H)$ -sentence, and let $\iota(\sigma)$ be the $\mathcal{L}_{\text{pow}}(\mathbb{T})$ sentence obtained by replacing all parameters in σ with their image under ι . The following are equivalent:*

- (1) $M \models \sigma$ for some maximal Hardy field $M \supseteq H$;
- (2) $M \models \sigma$ for every maximal Hardy field $M \supseteq H$;
- (3) $\mathbb{T}_{\text{pow}} \models \iota(\sigma)$.

Proof. We fix a maximal Hardy field $M \supseteq H$, and we view M as a model of $T_{\text{pow},\text{sm}}^{\text{nl}}$. We need to show that $M \models \sigma$ if and only if $\mathbb{T}_{\text{pow}} \models \iota(\sigma)$. If H is ω -free, then this follows from Theorem 5.4. If H is grounded, then we may use Lemma 4.8 to replace H by an ω -free H_{pow} -field $H_{\omega,\text{pow}}$, thereby reducing to the case that H is ω -free. If $H = \mathbb{R}$, then this follows from Corollary 5.2: take an $|M|^+$ -saturated elementary H_{pow} -field extension $L \supseteq \mathbb{T}_{\text{pow}}$ and extend $\iota: \mathbb{R} \rightarrow L$ to an H_{pow} -field embedding $M \rightarrow L$. This embedding is elementary, since $T_{\text{pow}}^{\text{nl}}$ is model complete, so M and L (and therefore M and \mathbb{T}_{pow}) have the same $\mathcal{L}_{\text{pow}}(H)$ -theory.

Suppose now that $H \supsetneq \mathbb{R}$ and that H is ungrounded. We will show that H must be ω -free (so the above cases are exhaustive). As \mathbb{T}_{pow} is ω -free, it is enough by Fact 1.3 to show that $\iota(\Gamma_H^<)$ is cofinal in $\Gamma_{\mathbb{T}}^<$ (where we write $\iota: \Gamma_H \rightarrow \Gamma_{\mathbb{T}}$ for the map induced by $\iota: H \rightarrow \mathbb{T}_{\text{pow}}$). Let $f \in H^>$ with $vf < 0$, and put $g := \iota(f)$. Using the fact that H is ungrounded, we define a sequence $(\gamma_n)_{n \in \mathbb{N}}$ from $\Gamma_H^<$ as follows:

$$\gamma_0 := vf, \quad \gamma'_{n+1} = \gamma_n^\dagger$$

Then $v(\log_n g) = \iota(\gamma_n)$ for each n , where $\log_n g$ is the n -fold iterated logarithm of g . It is well-known that $(\log_n g)$ is coinital among the positive infinite elements of \mathbb{T} , so $(\iota(\gamma_n))_{n \in \mathbb{N}}$ is cofinal in $\Gamma_{\mathbb{T}}^<$. \square

Our proof of Theorem 6.4 relies on the fact that any H -subfield of \mathbb{T} that properly contains \mathbb{R} is either grounded or ω -free, a dichotomy that does not apply to all Hardy fields. Given a real closed Hardy field $H \supseteq \mathbb{R}$ that is closed under powers, one may ask whether (1) and (2) in Theorem 6.4 are always equivalent, even when H doesn't embed into \mathbb{T}_{pow} . That is, do all maximal Hardy field extensions of H have the same $\mathcal{L}_{\text{pow}}(H)$ -theory? With a bit of care, one can show that this is indeed the case, arguing along the lines of [5, Theorem 12.3]. This uses the formalism of $\Lambda\Omega$ -fields from [3, Chapter 16] and the key fact that Hardy fields admit canonical $\Lambda\Omega$ -expansions; see [5, Lemma 12.1]. Explicitly, one needs to show that the various $\Lambda\Omega$ -field extensions used in proving [3, Proposition 16.4.1] can be carried out with a constant power map around.

6.2. Surreal numbers. The field \mathbf{No} of surreal numbers is a real closed field extension of \mathbb{R} introduced by Conway [8]. The surreals may be defined in several equivalent ways, but for our purposes, we define a surreal number to be a map $a: \gamma \rightarrow \{-, +\}$, where γ is an ordinal. For such a , the ordinal γ is called the **length of a** (sometimes called the *tree-rank* or *birthday* of a , depending on which definition of the surreals is being used). The collection of all surreal numbers is a proper class, and each ordinal γ is identified with the surreal number of length γ that takes constant value $+$. For each γ , we let $\mathbf{No}(\gamma)$ be the set of surreal numbers of length $< \gamma$.

The surreals admit an exponential, defined by Kruskal and Gonshor [14], and with this exponential, \mathbf{No} is an elementary extension of \mathbb{R}_{exp} [10]. More recently, Berarducci and Mantova equipped the surreals with a derivation that makes \mathbf{No} a Liouville closed H -field with constant field \mathbb{R} and satisfies the identity $\partial \exp(a) = \exp(a)\partial a$. We let \mathbf{No}_{pow} denote the expansion of \mathbf{No} by the Berarducci–Mantova derivation and the constant power map $(f, r) \mapsto \exp(r \log f): \mathbf{No}^> \times \mathbb{R} \rightarrow \mathbf{No}^>$, so $\mathbf{No}_{\text{pow}} \models T_{\text{pow}}$ by Proposition 3.3. The H -field \mathbf{No} was shown to be ω -free and newtonian in [4], so \mathbf{No}_{pow} is even a model of $T_{\text{pow},\text{sm}}^{\text{nl}}$.

Let κ be a regular uncountable cardinal. Then the set $\mathbf{No}(\kappa)$ is an ω -free newtonian Liouville closed H -subfield of \mathbf{No} containing \mathbb{R} [4, Corollary 4.6]. Lemma 3.1 gives that $\mathbf{No}(\kappa)$ is an elementary \mathcal{L}_{pow} -substructure of \mathbf{No}_{pow} . The next proposition, an analog of [4, Theorem 3], shows that the surreal numbers are *universal* among models of T_{pow} with small derivation and archimedean constant field.

Proposition 6.5. *Let K be a set-sized H_{pow} -field with small derivation and archimedean constant field. Then K admits an H_{pow} -field embedding into \mathbf{No}_{pow} .*

Proof. It suffices to show that some H_{pow} -field extension of K admits an H_{pow} -field embedding into \mathbf{No}_{pow} . Since K has small derivation, either $0 \in \Psi^\downarrow$ or 0 is a gap in K . In the case that 0 is a gap, we can use Lemma 4.7 with $s = 1$ to extend K to a grounded H_{pow} -field M with $0 \in (\Gamma_M^<)' \subseteq \Psi_M^\downarrow$ and $C_M = C$. Replacing K with M , we may assume that $0 \in \Psi^\downarrow$, so any H_{pow} -field extension of K has small derivation. By Corollary 4.9 and Proposition 4.3, K has a Liouville closed ω -free newtonian H_{pow} -field extension with the same constant field as K , so we may assume that $K \models T_{\text{pow},\text{sm}}^{\text{nl}}$. Let $\kappa := |K|^+$. Then $\mathbf{No}(\kappa) \models T_{\text{pow},\text{sm}}^{\text{nl}}$ and by [4, Lemma 5.3], the underlying ordered sets of $\mathbf{No}(\kappa)$ and $\Gamma_{\mathbf{No}(\kappa)}$ are κ -saturated. As C with its induced exponential is an archimedean model of $\text{Th}(\mathbb{R}, x \mapsto 2^x)$, we get an elementary exponential field embedding $(C, c \mapsto 2^c) \rightarrow (\mathbb{R}, x \mapsto 2^x)$ by the Laskowski–Steinhorn theorem [19]. By Corollary 5.2 with $\mathbf{No}(\kappa)$ in place of L , this embedding extends to an H_{pow} -field embedding $K \rightarrow \mathbf{No}(\kappa)$. \square

Corollary 6.6. *Every real closed Hardy field $H \supseteq \mathbb{R}$ that is closed under powers admits an H_{pow} -field embedding into the surreal numbers over \mathbb{R} .*

There is a natural ordered exponential field embedding $\iota: \mathbb{T} \rightarrow \mathbf{No}$, which was shown to be an elementary H -field embedding in [4]. As ι respects the constant power map, it is even an elementary H_{pow} -field embedding $\mathbb{T}_{\text{pow}} \rightarrow \mathbf{No}_{\text{pow}}$.

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