DIMENSION THEORY FOR THE ASYMPTOTIC COUPLE OF THE FIELD OF LOGARITHMIC TRANSSERIES

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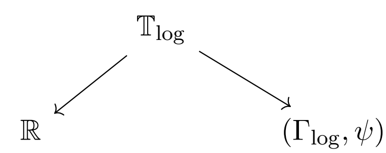
Abstract. In this paper we completely characterize all dimension functions on all models of the theory $T_{\log}$ of the asymptotic couple of the field of logarithmic transseries (Dimension Theorem). This is done by characterizing the "small" 1-variable definable sets (Small Sets Theorem). As a byproduct, we show that $T_{\log}$ is d-minimal and does not eliminate imaginaries. Separately, we provide an abstract criterion for d-minimality, which we use to observe some new examples of d-minimal expansions of valued fields.

1. Introduction

The differential field $\mathbb{T}_{\log}$ of logarithmic transseries is conjectured to have nice model-theoretic properties [25]. As $\mathbb{T}_{\log}$ is a so-called *H-field* [3], it is an expansion of a *valued* differential field; as such, in the Ax–Kochen–Ershov (AKE) tradition we view $\mathbb{T}_{\log}$ in terms of the "decomposition":



Here, $\mathbb{R}$ is simultaneously the residue field of the valuation and the constant field of the derivation (conjectured to have semialgebraic induced structure). The object $\Gamma_{\log}$ is the value group of the valuation, further equipped with a map $\psi\colon \Gamma_{\log} \to \Gamma_{\log} \cup \{\infty\}$ induced by the logarithmic derivative of $\mathbb{T}_{\log}$. Collectively, the pair $(\Gamma_{\log}, \psi)$ is called the *asymptotic couple* of $\mathbb{T}_{\log}$.

Incidentally, the object $(\Gamma_{\log}, \psi)$ is also the asymptotic couple of the Hardy field $\mathbb{R}(\mathfrak{L})$, i.e., the Hardy field over $\mathbb{R}$ which is generated by all power-products of the form $x^{r_0}(\log x)^{r_1}(\log^{\circ 2} x)^{r_2}\cdots(\log^{\circ n} x)^{r_n}$ for varying n, with each $r_i \in \mathbb{R}$. As pointed out in [18]: "many functions in number theory are comparable, or are conjectured to be comparable" to a function in $\mathbb{R}(\mathfrak{L})$. We contend that the asymptotic couple $(\Gamma_{\log}, \psi)$ is an appropriate universal domain for capturing the generic asymptotic behavior of the functions in $\mathbb{R}(\mathfrak{L})$, as well as their interaction with the (logarithmic) derivative.

This paper is the fourth in a series [23, 24, 26] about the theory $T_{\log}$ (defined in Section 2) of the asymptotic couple $(\Gamma_{\log}, \psi)$. To summarize, here are the most relevant things already known:

- $T_{\log}$ has quantifier elimination (QE) and a universal axiomatization (UA) in a natural language [23]; in particular, the quantifier-free definable sets enjoy a Tarski–Seidenberg-type theorem, and definable functions are given piecewise by terms in the language.
- $T_{\log}$ has the non-independence property (NIP) [24]; in particular, this implies definable hypothesis spaces are subject to the so-called *Fundamental Theorem of Statistical Learning*, which tells us they are always *PAC learnable* (see the surveys [9, 33]). Moreover, $T_{\log}$ is not strongly dependent and so it is not dp-minimal nor does it have finite dp-rank [26].
- $T_{\log}$ is distal [26]; in particular, definable relations satisfy strong combinatorial bounds including a definable version of the *strong Erdős–Hajnal property* [11, 10].

- The model-theoretic algebraic closure acl is in general *not* a pregeometry [24]; in particular, there is no immediate off-the-shelf *dimension theory* we can use similar to the likes of *vector spaces*, *algebraically closed fields*, and *o-minimal structures*.

In this paper, we further examine the nature of definable sets in models of $T_{\log}$ from topological, algebraic, and model-theoretic perspectives. Specifically, we answer the following questions:

Question. What are the dimension functions on models of $T_{\log}$?

Answer. The Dimension Theorem 1.1 completely characterizes all dimension functions on models of $T_{\log}$, where *dimension function* is meant in the axiomatic sense of [16]; see Definition 3.1. Corollary 3.16 shows that each such dimension coincides with an appropriate *topological dimension*.

Question. What is a more precise description of the 1-variable definable subsets of models of $T_{\log}$?

Answer. The Small Sets Theorem 3.11 characterizes the ideal of "small" unary definable sets (i.e., sets of dimension ≤ 0) for each dimension function, and forms the technical core of this paper.

Question. What is a more precise description of the n-variable definable subsets of models of $T_{\log}$?

Answer. This is the Kinda Small Sets Theorem 3.15 and the Very Small Sets Theorem 3.22. When $n > 1$, the conditions in the Small Sets Theorem 3.11 generalize in two ways: they either characterize sets of dimension $< n$ ("kinda small sets") or sets of dimension ≤ 0 ("very small sets").

Question. What is a more precise description of definable functions in models of $T_{\log}$?

Answer. Corollary 3.17 asserts that every definable function $f\colon \Gamma^n \to \Gamma_\infty$ is either locally affine or locally constant outside of a "kinda small set". Conversely, Proposition 4.17 characterizes functions defined on the "typical very small set" Ψ^n. Understanding these two extremes is sufficient for many questions about definable functions due to the inductive nature of dimension.

Question. Is the theory $T_{\log}$ d-minimal?

Answer. Yes, i.e., every 1-variable definable set in any model either has nonempty interior or is a finite union of discrete sets (Definition 5.2). First, this is apparent from (2)$\Leftrightarrow$(3) of the Small Sets Theorem 3.11. A second proof follows readily from a general *d-minimality criterion*, Proposition 5.5. Finally, Corollary 3.25 shows that $T_{\log}$ is d-minimal in the stronger sense of [20, Definition 9.1], which places additional topological requirements on the definable sets in n-variables; this relies on the Kinda Small and Very Small Sets Theorems 3.15 and 3.22.

Question. Does the 1-sorted theory $T_{\log}$ have elimination of imaginaries (EI)?

Answer. No. The analogue of the *RV sort* cannot be eliminated (Lemma 3.29). This uses properties of dimension and a cardinality argument in what we call *the standard model*.

Overview and main ideas. Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\bigoplus_n \mathbb{R}e_n$ be a vector space over $\mathbb{R}$ with basis (e_n). Then $\bigoplus_n \mathbb{R}e_n$ can be made into an ordered group using the usual lexicographical order, i.e., by requiring for nonzero $\sum_i r_i e_i$, that

$$\sum r_i e_i > 0 \quad \Longleftrightarrow \quad r_n > 0 \text{ for the least } n \text{ such that } r_n \neq 0.$$

Let $\Gamma_{\log}$ be the above ordered abelian group $\bigoplus_n \mathbb{R}e_n$. It is convenient to think of an element $\sum r_i e_i$ as the vector $(r_0, r_1, r_2, \ldots)$. We follow Rosenlicht [36] in taking the function:

$$\psi\colon \Gamma_{\log} \setminus \{0\} \to \Gamma_{\log}, \quad (\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}, \ldots) \mapsto (\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots)$$

as a new primitive, calling the pair $(\Gamma_{\log}, \psi)$ an *asymptotic couple*. Throughout, we refer to this specific asymptotic couple, depicted in Figure 1, as *the standard model*.

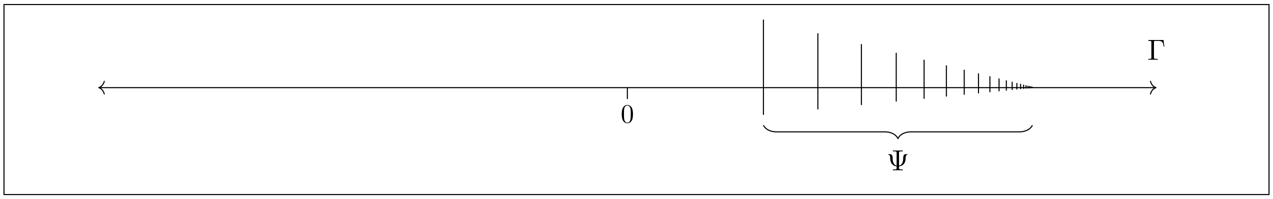


Figure 1: The standard model of $T_{\log}$.

The first key observation about the function ψ is that it is a convex valuation on the ordered abelian group $\Gamma_{\log}$. Moreover, the *value set* Ψ of ψ is a very important definable subset of $\Gamma_{\log}$:

$$\Psi \;:=\; \psi(\Gamma_{\log}\setminus\{0\}) \;=\; \{(\underbrace{1,\ldots,1}_{n},0,0,\ldots) : n\geq 1\}.$$

Note that this situation is a bit atypical in valuation theory, as usually a value set (or value group) of a valuation lives on its own more primitive sort, and is not a subset of the domain of the valuation. Furthermore, Ψ introduces a discrete set into the otherwise "continuous" object $\Gamma_{\log}$, which itself is o-minimal as an ordered divisible abelian group. In general, these seem to be the main sources of complications when dealing with $(\Gamma_{\log},\psi)$.

The story of dimension now begins with the following observation:

> Although acl is not a pregeometry in general, the *relativization* of acl to the definable set Ψ is a pregeometry. Moreover, this relativization $X\mapsto \operatorname{acl}(X\cup\Psi)$ is essentially the same as the "linear" pregeometry $X\mapsto \operatorname{span}_{\mathbb{Q}}(X\cup\Psi)$ coming from the underlying divisible abelian group structure of $\Gamma_{\log}$.

This pregeometry gives rise to a *dimension function*. However, the following issue still remains:

> *How many* dimension functions does a model of $T_{\log}$ have? Since there is no obvious definable field structure, uniqueness results such as [20, Theorem 3.48] do not apply.

The general role of *coarsening* in the analysis of H-fields [3] provides a natural guess at what the other dimension functions might be: they are parametrized by a certain "scale" that is uniformly indexed by the Ψ-set of the asymptotic couple, which we now explain.

For an arbitrary model (Γ,ψ) of $T_{\log}$ we set $\Psi := \psi(\Gamma^{\neq})$, where $\Gamma^{\neq} := \Gamma\setminus\{0\}$. Then for $\phi\in\Psi$ we define a proper convex subgroup of Γ:

$$\Delta_{\phi} \;:=\; \{x\in\Gamma^{\neq} : \psi(x)>\phi\}\cup\{0\}.$$

We extend this to $\phi\in\Psi\cup\{\infty\}$ by setting $\Delta_{\infty} := \{0\}$. We may then proceed to study a definable set $X\subseteq\Gamma$ in terms of its image in the quotient Γ/Δ_{ϕ}, which we visualize in Figure 2.

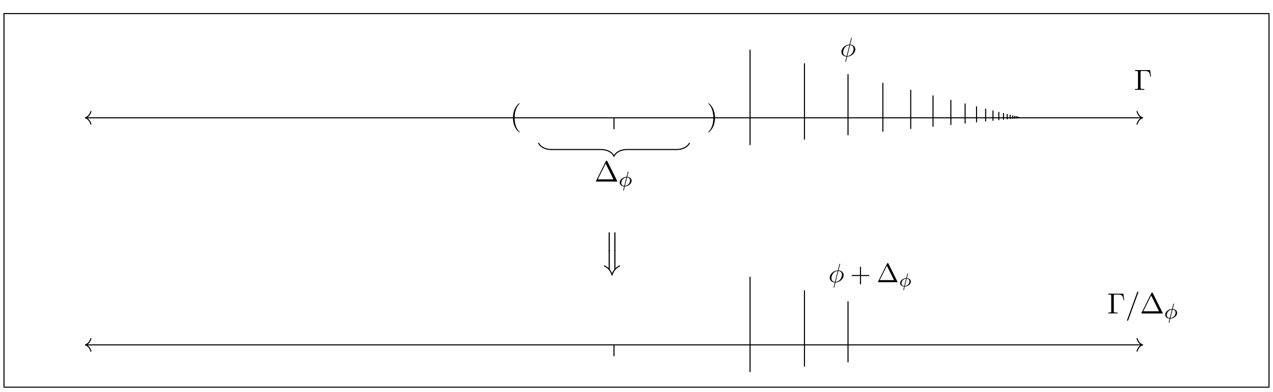


Figure 2: Quotienting by Δ_{ϕ}.

To do this, we consider the further relativization $X \mapsto \mathrm{acl}(X \cup \Psi \cup \Delta_\phi)$, which is also a pregeometry and yields a notion of *dimension* $\dim_\phi$ on definable sets. However, it is not obvious that $\dim_\phi$ is a *dimension function* in the axiomatic sense of [16] (indeed, not every pregeometric dimension is: the theory of $(\mathbb{N};<)$ is pregeometric, but the corresponding dimension is not definable). Our main theorem establishes that the $\dim_\phi$ are dimension functions, and they are the only ones:

Dimension Theorem 1.1. *Suppose* (Γ, ψ) *is a model of* $T_{\log}$*. Then:*

- *(1) for each* $\phi \in \Psi \cup \{\infty\}$*, there exists a unique dimension function* $\dim_\phi$ *on* (Γ, ψ) *such that* $\dim_\phi \Delta_\phi = 0$ *and* $\dim_\phi \Delta_\xi = 1$ *for all* $\xi \in \Psi^{<\phi}$*, and*
- *(2) if* d *is an arbitrary dimension function on* (Γ, ψ)*, then* $\mathrm{d} = \dim_\phi$ *for some* $\phi \in \Psi \cup \{\infty\}$*.*

This theorem follows from the Small Sets Theorem 3.11, which characterizes the definable sets $X \subseteq \Gamma$ such that $\dim_\phi X \leq 0$. Part of this characterization includes the fact that $T_{\log}$ is d-minimal.

At an earlier stage of this work, we also produced a direct proof that $T_{\log}$ is d-minimal by extracting and applying a d-minimality criterion, Proposition 5.5, from other proofs of d-minimality. This proof was later superseded by the finer analysis needed for the Small Sets Theorem 3.11. Nevertheless, we have included this criterion because it may be useful for other topological theories. For example, we use it to show in Subsection 5.3 that henselian valued fields of equicharacteristic zero, equipped with a section of the valuation and a lift of the residue field, are d-minimal. See Section 5 for a self-contained treatment of this criterion and other applications.

1.1. **Outline of paper.** In Section 2 we review the theory $T_{\log}$ and establish some basic facts we need. Section 3 contains the main results of the paper; see the introduction of that section for an overview. Section 4 is dedicated to the proof of the Small Sets Theorem; in particular, Subsection 4.1 provides a roadmap of the proof and the rest of the section consists of independent subsections contributing some part of the proof. Section 5 provides an abstract criterion for d-minimality for topological theories expanded by unary functions, along with several applications. In Section 6 we make some final comments and observations. Finally, in Appendix A we collect a few basic topological facts and definitions that we use and prove some facts needed for Section 5.

1.2. **Conventions.**

Set theory conventions. Given a boolean algebra $\mathcal{C}$ of subsets of a set X, recall that an **ideal** of $\mathcal{C}$ is a collection $\mathcal{I} \subseteq \mathcal{C}$ such that $\varnothing \in \mathcal{I}$, if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$, and if $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$. Given a set $X \subseteq A \times B$ and $a \in A$, $X_a \coloneqq \{b \in B : (a,b) \in X\}$ denotes the *fiber* of X over a. We may use $\sqcup$ and $\bigsqcup$ instead of $\cup$ and $\bigcup$ to emphasize that a given union is a disjoint union.

Ordered set conventions. By "ordered set" we mean "totally ordered set". Let S be an ordered set. Below, the ordering on S will be denoted by $\leq$, and a subset of S is viewed as ordered by the induced ordering. Suppose that B is a subset of S. We put $S^{>B} \coloneqq \{s \in S : s > b \text{ for every } b \in B\}$ and $S^{>a} \coloneqq S^{>\{a\}}$; similarly for $\geq$, $<$, and $\leq$ instead of $>$. For $a, b \in S$ we put

$$[a,b]_B \ \coloneqq\ \{x \in B : a \leq x \leq b\}.$$

If $B = S$, then we usually write $[a,b]$ instead of $[a,b]_S$. A subset C of B is said to be **convex** in B if for all $a, b \in C$ we have $[a,b]_B \subseteq C$. For $a < b$ we also set

$$(a,b) \ \coloneqq\ \{x \in X : a < x < b\}.$$

The sets of the form (a,b) are called **intervals** in S. The intervals form a basis for a hausdorff topology on S, the **order topology** on S.

Ordered abelian group conventions. Suppose that G is an ordered abelian group. Then we set $G^{\neq} := G\setminus\{0\}$. Also $G^{<} := G^{<0}$; similarly for $\geq$, $\leq$, and $>$ instead of $<$. We define $|g| := \max(g, -g)$ for $g \in G$. For $a \in G$, the **archimedean class** of a is defined by

$$[a] \ := \ \{g \in G : |a| \leq n|g| \text{ and } |g| \leq n|a| \text{ for some } n \geq 1\}.$$

Basic facts about ordered abelian groups and archimedean classes may be found in [3, 2.4].

Topology conventions. Given a subset A of a topological space X, we let A' denote the **derived set** of A (in X), i.e., the set of limit points of A in X. For $n > 0$, we set $A^{(n)} := (A^{(n-1)})'$, where $A^{(0)} := A$. We say that A is **d-finite** (in X) if $A^{(n)} = \varnothing$ for some n. Given a function $f\colon X \to Y$ between topological spaces X, Y, we let $\mathsf{Discont}(f)$ denote the set of points of X at which f is discontinuous. We let $\operatorname{br} A := A \setminus \operatorname{int} A$ denote the **border** of A. See Appendix A for more on d-finite sets, the border of A, and other topological facts we use.

Model theory conventions. In general we adopt the conventions of [3, Appendix B]. In Sections 2, 3 and 4 we work in a 1-sorted setting, whereas in Section 5 we work in a possibly many-sorted setting.

Throughout, *A-definable* has its usual meaning, whereas *definable* means *definable with arbitrary parameters* (as opposed to meaning *$\varnothing$-definable*). In general we choose not to consider a monster model (except in Lemma 3.10) and instead opt to work in arbitrary models of a (complete) theory, taking (sufficiently saturated) elementary extensions as needed.

Given an elementary extension $\boldsymbol{M} \preccurlyeq \boldsymbol{M}^*$, and a definable set X in $\boldsymbol{M}$, it is understood that X^* denotes its realization in $\boldsymbol{M}^*$; moreover, given a tuple α in M^*, we regard $\operatorname{tp}(\alpha|M)$ as an ultrafilter on a boolean algebra of M-definable subsets, as opposed to a collection of formulas.

As is standard when working with pregeometries, we often use concatenation to denote a union of sets, e.g., $A\Psi\Delta_\phi := A \cup \Psi \cup \Delta_\phi$. We refer to [39, C.1] for the basic properties of pregeometries.

2. Overview of $T_{\log}$

In this section we provide an overview of the theory $T_{\log}$ and establish a few basic facts and conventions needed for the remainder of the paper. Since every asymptotic couple we will encounter will be a model of $T_{\log}$, we forgo a systematic development and instead define $T_{\log}$ directly from the standard model introduced in Section 1. See [3, Sections 6.5, 9.2] for a general treatment of asymptotic couples (including the definition), and [23] for an axiomatic treatment of $T_{\log}$.

2.1. More functions on the standard model.

First, since the ordered abelian group $\Gamma_{\log}$ is divisible, for $n \geq 1$ we may define:

$$\delta_n\colon \Gamma_{\log} \to \Gamma_{\log}, \quad \alpha \ \mapsto \ \delta_n(\alpha) \ := \ \tfrac{1}{n}\alpha.$$

Next, observe that $(\Gamma_{\log}, \psi)$ has *asymptotic integration*, i.e., for every $\alpha \in \Gamma_{\log}$ there exists a unique $\beta \in \Gamma_{\log}^{\neq}$ such that $\beta + \psi(\beta) = \alpha$. Thus, we may define the *asymptotic integral*:

$$\textstyle\int\colon \Gamma_{\log} \to \Gamma_{\log}^{\neq}, \quad \alpha \ \mapsto \ \int \alpha \ := \ \text{the unique } \beta \text{ such that } \beta + \psi(\beta) = \alpha.$$

Note that $\int$ is an increasing bijection. Next we define the *successor function*:

$$s\colon \Gamma_{\log} \to \Psi, \quad \alpha \ \mapsto \ s(\alpha) \ := \ \psi(\textstyle\int \alpha).$$

Note that $s0$ is the smallest element of Ψ. For each $\phi \in \Psi$, the element $s\phi$ is the immediate successor of ϕ in the ordered set $(\Psi; <)$, so s restricts to an increasing bijection $s\colon \Psi \to \Psi^{>s0}$. Thus, we may define the *predecessor function*:

$$p\colon \Psi^{>s0} \to \Psi, \quad \phi \ \mapsto \ p(\phi) \ := \ \text{the unique } \xi \in \Psi \text{ such that } s\xi = \phi.$$

Finally, we adjoin to the underlying set $\Gamma_{\log}$ a new element ∞ and extend the ordering on $\Gamma_{\log}$ to $\Gamma_{\log} \cup \{\infty\}$ by declaring $\Gamma_{\log} < \infty$. Likewise, we extend the domains of the functions defined so far for $\Gamma_{\log}$ by declaring ∞ to be a default value, i.e., for every $n \geq 1$, $\alpha \in \Gamma_{\log} \cup \{\infty\}$, and $\beta \in (\Gamma_{\log} \cup \{\infty\}) \setminus \Psi^{>s0}$ we define:

$$-\infty \;=\; \alpha + \infty \;=\; \infty + \alpha \;=\; \psi(0) \;=\; \psi(\infty) \;=\; \delta_n(\infty) \;=\; \textstyle\int \infty \;=\; s(\infty) \;=\; p(\beta) \;\coloneqq\; \infty.$$

2.2. **The $\mathcal{L}_{\log}$-theory $T_{\log}$.** We construe the standard model $(\Gamma_{\log}, \psi)$ as an $\mathcal{L}_{\log}$-structure, where

$$\mathcal{L}_{\log} \;\coloneqq\; \{0, -, +, <, \psi, \infty, (\delta_n)_{n\geq 1}, s, p\}$$

and define $T_{\log} \coloneqq \mathrm{Th}_{\mathcal{L}_{\log}}(\Gamma_{\log}, \psi)$. By convention, we will always denote a model of $T_{\log}$ by a pair (Γ, ψ), where it is understood that:

- $\Gamma = (\Gamma; 0, -, +, <, (\delta_n)_{n\geq 1})$ is an ordered divisible abelian group, which we also regard as an ordered $\mathbb{Q}$-vector space;
- ψ is a function $\Gamma^{\neq} \to \Gamma$;
- the primitives s, p are left implicit as they can be defined in terms of ψ;
- the underlying set of the $\mathcal{L}_{\log}$-structure is $\Gamma_\infty \coloneqq \Gamma \cup \{\infty\}$, where $\Gamma < \infty$ and all primitives may be regarded as total functions;
- although not part of the language, we can make use of the definable function $\int : \Gamma \to \Gamma^{\neq}$, defined in the same way as for $(\Gamma_{\log}, \psi)$.

Remark 2.1 (Disclaimer about ∞)**.** As ∞ is an element of our underlying structure, we treat it as such in model-theoretic statements. However, most of our statements are made only considering Γ^n (without ∞). In this way we can use $\mathbb{Q}$-vector space arguments without having to accommodate ∞. Such statements can be readily adapted to Γ^n_∞.

For the rest of Section 2, we assume (Γ, ψ) is a model of $T_{\log}$.

We need the following identities, established in [23, Lemmas 3.2, 3.7, and 3.4]. The first connects s and $\int$ and implies we may regard $\int$ as an $\mathcal{L}_{\log}$-term:

Fact 2.2. *For all $\alpha, \beta \in \Gamma$:*

Integral Identity. $\int \alpha = \alpha - s\alpha$.
Fixed Point Identity. $\beta = \psi(\alpha - \beta)$ *if and only if* $\beta = s\alpha$.
Successor Identity. If $s\alpha < s\beta$, *then* $\psi(\beta - \alpha) = s\alpha$.

Fact 2.3 (QE [23, Theorem 5.2] and UA [23, Lemma 5.1])**.** *$T_{\log}$ has quantifier elimination (QE) and a universal axiomatization (UA).*

Here is an immediate consequence of QE and UA:

Lemma 2.4. *Suppose $(\Gamma_1, \psi_1) \models T_{\log}$ and $(\Gamma_0, \psi_0) \subseteq (\Gamma_1, \psi_1)$ is an $\mathcal{L}_{\log}$-substructure. Then $(\Gamma_0, \psi_0) \preccurlyeq (\Gamma_1, \psi_1)$, and thus $\psi_0(\Gamma_0^{\neq}) = \Gamma_0 \cap \psi_1(\Gamma_1^{\neq})$.*

Given a tuple α in an extension $(\Gamma_1, \psi_1) \succcurlyeq (\Gamma, \psi)$, we let $\Gamma\langle\alpha\rangle$ denote the $\mathcal{L}_{\log}$-substructure of Γ_1 generated by Γ and α. Then $(\Gamma, \psi) \preccurlyeq (\Gamma\langle\alpha\rangle, \psi_1|_{\Gamma\langle\alpha\rangle}) \preccurlyeq (\Gamma_1, \psi_1)$ by the previous lemma.

2.3. **The Ψ-set of a model.** Given a model (Γ_0, ψ_0) of $T_{\log}$, we set $\Psi_{\Gamma_0} \coloneqq \psi(\Gamma_0^{\neq})$, which we refer to as the *Ψ-set* of (Γ_0, ψ_0). Thus by Lemma 2.4:

$$\Psi_{\Gamma_0} \;=\; \Gamma_0 \cap \Psi_{\Gamma_1} \quad \text{for any model } (\Gamma_1, \psi_1) \text{ of } T_{\log} \text{ extending } (\Gamma_0, \psi_0).$$

When a distinguished model (Γ, ψ) is clear from context (as is currently the case), we will denote Ψ_Γ as just Ψ. Finally, we set $\Psi_\infty \coloneqq \Psi \cup \{\infty\}$.

Lemma 2.5. *Let Γ_1 be a $\mathbb{Q}$-linear subspace of Γ with $\Psi \subseteq \Gamma_1$. Then $(\Gamma_1, \psi|_{\Gamma_1})$ is an elementary substructure of (Γ, ψ) with $\Psi_{\Gamma_1} = \Psi$.*

Proof. From $\Psi \subseteq \Gamma_1$, we get that $\Gamma_{1,\infty}$ is closed under the functions ψ, s, and p. Hence $(\Gamma_1, \psi|_{\Gamma_1})$ is an $\mathcal{L}_{\log}$-substructure of (Γ, ψ). The claim now follows by Lemma 2.4. □

Observe that $(\Psi; <) \equiv (\mathbb{N}; <)$. In fact, this entirely characterizes the structure of the Ψ-set:

Fact 2.6 ([23, Corollary 7.2])**.** *If $(\Gamma, \psi) \models T_{\log}$, the structure $(\Psi; <)$ is purely stably embedded in (Γ, ψ) in the sense that the structure induced on Ψ by (Γ, ψ) is just its structure as an ordered set.*

Lemma 2.7. *If $C \subseteq \Psi$ is a definable convex subset of Ψ, then C is either of the form $[\alpha, \beta]_\Psi$ or $\Psi^{\geq \alpha}$ for some $\alpha, \beta \in \Psi$.*

Proof. This follows from Fact 2.6, and can also be seen directly by noting that "proper s-cuts" (see [24, 2.7]) are not definable since this would violate the universal property of the extension lemma [23, 4.12]. □

Lemma 2.8 (Locally constant primitives)**.** *When Γ is equipped with the order topology we have:*

(1) p takes constant value ∞ on $\Gamma \setminus \Psi^{>s0}$,
(2) ψ is locally constant on $\Gamma^{\neq}$,
(3) s is locally constant on Γ.

Proof. (1) Clear from the definition. (2) Suppose $x \neq 0$, say $x > 0$. Then $\psi(x/2) = \psi(2x)$, and so ψ takes constant value $\psi(x)$ on the interval $(x/2, 2x)$ since ψ is a convex valuation. (3) Note that $\int : \Gamma \to \Gamma^{\neq}$ is a homeomorphism since it is a strictly increasing bijection. Thus, the composition $s = \psi \circ \int$ is locally constant by (2). □

2.4. **Quotienting by Δ_ϕ.** *In this subsection we let ϕ range over Ψ_∞.* Recall:

Definition 2.9. For $\phi \in \Psi$ define the proper convex subgroup of Γ:

$$\Delta_\phi \;\coloneqq\; \{x \in \Gamma : \psi(x) > \phi\}.$$

We extend this to $\phi \in \Psi_\infty$ by setting $\Delta_\infty \coloneqq \{0\}$.

Lemma 2.10. *The set of definable convex subgroups of Γ is $\{\Delta_\phi : \phi \in \Psi_\infty\} \cup \{\Gamma\}$.*

Proof. Suppose $\Delta \subseteq \Gamma$ is a definable convex subgroup and consider $X \coloneqq \{\phi \in \Psi : \Delta_\phi \supseteq \Delta\}$, a definable initial segment of Ψ; by Lemma 2.7 there are three cases to consider. If $X = \Psi$, then $\Delta = \{0\} = \Delta_\infty$. If $X \neq \Psi$ is nonempty, then for $\phi \coloneqq \max X$, we have $\Delta_\phi \supseteq \Delta \supsetneq \Delta_{s\phi}$. We claim $\Delta_\phi = \Delta$. Otherwise, since Δ is definable, this would contradict the universal property in the extension lemma [23, 4.6] since there would be two nonisomorphic ways of adding an element at the cut "Δ^+": one that adds a new archimedean class to the extension of the definable set Δ, and one that does not. Finally, if $X = \emptyset$, then we have $\Delta = \Gamma$ by a similar argument. □

Finally, we establish the following notation and conventions with regard to quotienting by Δ_ϕ:

- Given the subgroup $\Delta_\phi \subseteq \Gamma$, let $\pi_\phi \colon \Gamma \to \Gamma/\Delta_\phi$ denote the projection map. When ϕ is understood from context, we denote $\bar{\Gamma} \coloneqq \Gamma/\Delta_\phi$. Since Δ_ϕ is a convex $\mathbb{Q}$-subspace, we may construe $\bar{\Gamma}$ as an ordered $\mathbb{Q}$-vector space, with ordering induced by the ordering on Γ.
- Given the subgroup $\Delta_\phi^n \subseteq \Gamma^n$, we identify Γ^n/Δ_ϕ^n with $(\Gamma/\Delta_\phi)^n$, which we also denote by $\bar{\Gamma}^n$ when ϕ is understood from context. We also have the natural projection map

$$\pi_\phi \colon \Gamma^n \to \bar{\Gamma}^n, \quad (\alpha_1, \ldots, \alpha_n) \;\mapsto\; (\alpha_1 + \Delta_\phi, \ldots, \alpha_n + \Delta_\phi) \;=\; (\alpha_1, \ldots, \alpha_n) + \Delta_\phi^n.$$

- Given $\alpha \in \Gamma^n$ and $X \subseteq \Gamma^n$, we denote $\pi_\phi(\alpha)$ and $\pi_\phi(X)$ by just $\bar{\alpha}$ and $\bar{X}$ when ϕ is understood from context.

3. Dimension, the Small Sets Theorem, and related results

This section contains the main results of this paper. First, Subsection 3.1 reviews the definition and main properties of a *dimension function* on a structure, and makes some basic observations about how dimension functions on models of $T_{\log}$ must behave. Next, Subsection 3.2 introduces the family of pregeometries on models of $T_{\log}$ that will later give rise to the family of dimension functions. Subsection 3.3 contains the statement of the Small Sets Theorem 3.11 (proof deferred to Section 4). Subsection 3.4 shows how the Dimension Theorem 1.1 follows from (part of) the Small Sets Theorem. In Subsection 3.5 we see how the Small Sets Theorem splits into the Kinda Small Sets Theorem 3.15 and Very Small Sets Theorem 3.22 when considering definable sets of higher arity. We also observe some consequences, including local linearity (Corollary 3.17) and d-minimality (Corollary 3.25). Finally, Subsection 3.6 shows the failure of elimination of imaginaries, as a consequence of dimension theory.

In this section (Γ, ψ) ranges over models of $T_{\log}$, and ϕ ranges over Ψ_∞, where $\Psi = \psi(\Gamma^{\neq})$. We equip $\bar{\Gamma} = \Gamma/\Delta_\phi$ with the order topology and $\bar{\Gamma}^n = (\Gamma/\Delta_\phi)^n = \Gamma^n/\Delta_\phi^n$ with the product topology. Given a set $X \subseteq \Gamma^n$, we construe $\bar{X} = \pi_\phi(X)$ as a subset of the ambient topological space $\bar{\Gamma}^n$.

3.1. **Dimension functions.** We adopt the following definition of a *dimension function* from [16]; below we declare $-\infty < \mathbb{N}$ and set $n + (-\infty) := -\infty$ for every $n \in \mathbb{N}$.

Definition 3.1. Let $\boldsymbol{M}$ be a 1-sorted structure. A **dimension function** on $\boldsymbol{M}$ is a function d from the definable subsets of M^n (n varying) to $\mathbb{N} \cup \{-\infty\}$ such that:

- (D1) (a) $\mathrm{d}(S) = -\infty \Leftrightarrow S = \varnothing$ for definable $S \subseteq M^n$;
  - (b) $\mathrm{d}(\{a\}) = 0$ for all $a \in M$;
  - (c) $\mathrm{d}(M) = 1$;
- (D2) $\mathrm{d}(S_1 \cup S_2) = \max\{\mathrm{d}(S_1), \mathrm{d}(S_2)\}$ for definable $S_1, S_2 \subseteq M^n$;
- (D3) d is preserved under permutation of coordinates;
- (D4) if $S \subseteq M^{n+1}$ is definable, then $B_i := \{a \in M^n : \mathrm{d}(S_a) = i\}$ is definable and

$$\mathrm{d}\big(\{(a,b) \in S : a \in B_i\}\big) \;=\; i + \mathrm{d}(B_i) \quad \text{for } i = 0, 1.$$

These four axioms have several natural consequences. Here we collect a few that we need.

Fact 3.2 ([16, 1.1, 1.3, 1.5], [1, 2.2])**.** *Let $\boldsymbol{M}$ be a structure and* d *be a dimension function on $\boldsymbol{M}$.*

(1) If $S_1, S_2 \subseteq M^n$ are definable and $S_1 \subseteq S_2$, then $\mathrm{d}\, S_1 \leq \mathrm{d}\, S_2$.
(2) If $S \subseteq M^n$ is finite and nonempty, then $\mathrm{d}\, S = 0$.
(3) $\mathrm{d}\, M^n = n$.
(4) If d' is a dimension function on $\boldsymbol{M}$ such that $\mathrm{d}'\, S = \mathrm{d}\, S$ for all definable $S \subseteq M$, then $\mathrm{d}' = \mathrm{d}$.
(5) If $S \subseteq M^m$ and $f\colon S \to M^n$ are definable, then:
  - *(a) $\mathrm{d}(S \times T) = \mathrm{d}\, S + \mathrm{d}\, T$ for any definable $T \subseteq M^n$;*
  - *(b) $\mathrm{d}\, S \geq \mathrm{d}\, f(S)$ (in particular, $\mathrm{d}\, S = \mathrm{d}\, f(S)$ for injective f);*
  - *(c) $B_i := \{a \in M^n : \mathrm{d}\, f^{-1}(a) = i\}$ is definable and $\mathrm{d}\, f^{-1}(B_i) = i + \mathrm{d}\, B_i$ for $i = 0, \ldots, m$.*

(6) For $S \subseteq M^n$ definable and $d \leq n$, we have $\mathrm{d}\, S \geq d$ if and only if $\mathrm{d}\, \pi(S) = d$ for some coordinate projection $\pi\colon M^n \to M^d$.

Lemma 3.3. *Let* d *be a dimension function on an expansion of a linear order $(M; <, \ldots)$ and $X \subseteq M$ be a definable subset such that for every $a \in X$, we have $\mathrm{d}\, X^{\leq a} = 0$. Then $\mathrm{d}\, X = 0$.*

Proof. Clearly, $\mathrm{d}\, X \in \{0, 1\}$, so $\mathrm{d}\, X^2 \in \{0, 2\}$ by Fact 3.2(5a). However, by (D4) the "triangle" $\{(a,b) \in X^2 : b \leq a\}$ has dimension $\mathrm{d}\, X$ since the vertical fibers $X_a = X^{\leq a}$ have dimension 0. Since the square X^2 can be covered by two definably bijective copies of this triangle, it follows from Fact 3.2(5b) and (D2) that $\mathrm{d}\, X^2 \in \{0, 1\}$, hence $\mathrm{d}\, X^2 = 0$. Thus $\mathrm{d}\, X = 0$. □

Corollary 3.4. *Let* d *be a dimension function on* (Γ,ψ). *Then* $\mathrm{d}\,\Psi = 0$.

Proof. Consider $X \coloneqq \{\alpha \in \Psi : \mathrm{d}[s0,\alpha]_\Psi = 0\}$, which is a definable initial segment of Ψ by (D4) and Fact 3.2(1). By Lemma 3.3, it suffices to show that $X = \Psi$. By (D1) and (D2), $s0 \in X$ and if $\alpha \in X$, then $s\alpha \in X$. Hence $X = \Psi$ by Lemma 2.7, as desired. □

In an ordered abelian group, there is always a largest definable convex subgroup with dimension 0:

Lemma 3.5. *Let* d *be a dimension function on an expansion* $(G;<,+,\dots)$ *of an ordered abelian group. Then* G *has a largest definable convex subgroup* Δ *such that* $\mathrm{d}\,\Delta = 0$. *For an interval* $(a,b) \subseteq G$, *we have* $\mathrm{d}(a,b) = 1$ *if and only if* $b - a > \Delta$.

Proof. Define $\Delta \coloneqq \{x \in G : \mathrm{d}[-|x|,|x|] = 0\}$. The set Δ is convex by Fact 3.2(1) and definable by (D4). It follows by Fact 3.2(5b) and (D2) that if $x \in \Delta$, then $2x \in \Delta$, so Δ is a subgroup. Applying Lemma 3.3 to $\Delta^{\geq}$ yields $\mathrm{d}\,\Delta = 0$. The final statement follows by similar calculations. □

The following consequence of Lemmas 2.10 and 3.5 limits the possible dimension functions:

Corollary 3.6. *If* d *is a dimension function on* (Γ,ψ), *then there exists a unique* ϕ *such that* $\mathrm{d}\,\Delta_\phi = 0$ *and* $\mathrm{d}\,\Delta_\xi = 1$ *for every* $\xi \in \Psi^{<\phi}$; *moreover, for an interval* $(a,b) \subseteq \Gamma$, *we have* $\mathrm{d}(a,b) = 0$ *if and only if* $b - a \in \Delta_\phi$.

Corollary 3.6 notwithstanding, the following two things are still not clear at this point:

- (Uniqueness) For each ϕ, there is *at most one* dimension function d such that $\mathrm{d}\,\Delta_\phi = 0$ and $\mathrm{d}\,\Delta_\xi = 1$ for $\xi \in \Psi^{<\phi}$.
- (Existence) For each ϕ, there is *at least one* such dimension function.

Remark 3.7. If $\boldsymbol{R} = (R;<,+,\dots)$ is an o-minimal expansion of an ordered group, then Lemma 3.5 already provides the *uniqueness* of a dimension function as an easy consequence of the o-minimality axiom (it implies all intervals must have dimension 1), although *existence* of a dimension function requires more work (via the Cell Decomposition Theorem, or by proving acl is a pregeometry as a consequence of the Monotonicity Theorem).

3.2. **Pregeometries.** For each ϕ define a closure operator cl_ϕ on Γ_∞ by setting for each $A \subseteq \Gamma_\infty$:

$$\mathrm{cl}_\phi(A) \ \coloneqq\ \mathrm{acl}(A\Psi\Delta_\phi).$$

Although the model-theoretic algebraic closure acl in $T_{\log}$ is not a pregeometry, it follows from QE and UA that $\mathrm{cl}_\phi(A) \setminus \{\infty\} = \mathrm{span}_{\mathbb{Q}}(A\Psi\Delta_\phi) = \mathrm{span}_{\mathbb{Q}}(A\Psi) + \Delta_\phi$ for $A \subseteq \Gamma$, so cl_ϕ is a pregeometry; in particular, $\mathrm{cl}_\phi(A) = \mathrm{cl}_\infty(A) + \Delta_\phi$. Moreover, $(\mathrm{cl}_\phi(A), \psi|_{\mathrm{cl}_\phi(A)})$ is an elementary substructure of (Γ,ψ) with $\Psi_{\mathrm{cl}_\phi(A)} = \Psi$ by Lemma 2.5.

Each pregeometry yields a notion of dimension defined as follows. Let $\mathrm{rk}_\phi(B|A)$ be the size of a basis of $\mathrm{cl}_\phi(AB)$ over $\mathrm{cl}_\phi(A)$. Then for a definable $X \subseteq \Gamma_\infty^n$, set

$$\dim_\phi(X) \ \coloneqq\ \sup\{\mathrm{rk}_\phi(\{x_0,\dots,x_{n-1}\}|\Gamma) : (x_0,\dots,x_{n-1}) \in X^*\} \ \in\ \{-\infty,0,1,\dots,n\},$$

where $(\Gamma^*,\psi^*) \succcurlyeq (\Gamma,\psi)$ is $|\Gamma|^+$-saturated and rk_ϕ is computed using the pregeometry of (Γ^*,ψ^*). This is independent of the choice of $|\Gamma|^+$-saturated $(\Gamma^*,\psi^*) \succcurlyeq (\Gamma,\psi)$ (see [1, Section 2] for a general statement of this kind). Thus, we have a family of dimensions such that if $\phi \leq \xi \in \Psi_\infty$, then $\dim_\phi(X) \leq \dim_\xi(X)$ for all definable $X \subseteq \Gamma_\infty^n$.

These dimensions also fit into the framework of [1]: In every model of $T_{\log}$, the pregeometry cl_∞ is defined by the collection of $\mathcal{L}_{\log}$-formulas of the form

$$\exists x_0 \cdots x_{m-1} \ \Big(\sum_{i=0}^{m-1} q_i\psi(x_i) + \sum_{j=0}^{n-1} r_j y_j = u\Big),$$

where $q_i, r_j \in \mathbb{Q}$ for $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$. For $\phi \in \Psi$, the pregeometry cl_ϕ is defined in every model of $\mathrm{Th}_{\mathcal{L}_{\log} \cup \{\phi\}}(\Gamma, \psi)$ by the collection of $\mathcal{L}_{\log} \cup \{\phi\}$-formulas of the form

$$\exists x_0 \cdots x_{m-1} \exists z \ \big(\psi(z) > \phi \wedge \big(\sum_{i=0}^{m-1} q_i \psi(x_i) + \sum_{j=0}^{n-1} r_j y_j + z = u\big)\big).$$

It is not obvious that each $\dim_\phi$ is a dimension function on (Γ, ψ) in the sense of Definition 3.1, although (D1)(a), (D1)(b), (D2), and (D3) are easy.

Lemma 3.8. *For any $\phi \in \Psi_\infty$, we have $\dim_\phi(\Gamma) = 1$, i.e., $\dim_\phi$ satisfies (D1)(c).*

Proof. Let (Γ^*, ψ^*) be an elementary extension of (Γ, ψ) containing an element $\alpha > \Gamma$. We claim that $\Gamma\langle\alpha\rangle = \Gamma \oplus \mathbb{Q}\alpha$. It is enough to show that $\Gamma \oplus \mathbb{Q}\alpha$ is closed under ψ^*, s, and p. Consider an element $\gamma + q\alpha$, where $\gamma \in \Gamma$ and $q \in \mathbb{Q}^{\neq}$. Since $[\alpha] > [\Gamma^{\neq}]$, we have $\psi^*(\gamma + q\alpha) = \psi^*(\alpha) = s0$. In particular, $s0 = \psi^*(\gamma + q\alpha - s0)$, so the Fixed Point Identity (Fact 2.2) gives $s(\gamma + q\alpha) = s0$. We also see that $\gamma + q\alpha$ is not in $\psi^*((\Gamma \oplus \mathbb{Q}\alpha)^{\neq})$, so $p(\gamma + q\alpha) = \infty$.

Having established this claim, we see that

$$\Psi_{\Gamma\langle\alpha\rangle} = \Psi, \qquad \{\beta \in \Gamma\langle\alpha\rangle : \psi^*(\beta) > \phi\} = \Delta_\phi,$$

so $\mathrm{cl}_\phi(\varnothing)$, computed in $\Gamma\langle\alpha\rangle$, is contained in Γ. Thus $\mathrm{rk}_\phi(\alpha|\Gamma) = 1$, so $\dim_\phi(\Gamma) = 1$. □

In light of Lemma 3.8, to get that $\dim_\phi$ is a dimension function it remains to establish (D4), which we do in Corollary 3.13 using part of the Small Sets Theorem. As a first step, compactness and (D2) yield (1)⇔(5) of the Small Sets Theorem (or see [1, Lemma 2.3]).

Lemma 3.9. *Let $X \subseteq \Gamma$ be definable. Then $\dim_\phi X \leq 0$ if and only if X is covered by finitely many affine maps $\Psi^n \times \Delta_\phi \to \Gamma$.*

For the remainder of the paper, we call a definable subset $X \subseteq \Gamma$ **ϕ-small** if $\dim_\phi X \leq 0$; equivalently, X is ϕ-small if it is covered by finitely many affine maps $\Psi^n \times \Delta_\phi \to \Gamma$.

Connection to existential matroids. When working in a monster model, the pregeometries cl_ϕ fit into the more flexible framework of *existential matroids* from [20]. Although we only explicitly use that framework in the proof of Corollary 3.14 below (Corollary 3.14 is used in the proofs of (8)⇒(1) of the Small Sets Theorem and (6)⇒(1) of the Kinda Small Sets Theorem), it has helped us to understand pregeometries and dimension functions.

Lemma 3.10. *Suppose (Γ, ψ) is a monster model of $T_{\log}$, expanded to the language $\mathcal{L}_{\log} \cup \{\phi\}$. Then cl_ϕ is an existential matroid in the sense of [20].*

3.3. The Small Sets Theorem.

We first consider an example of a typical (∞-)small set.

Consider the definable set (pictured in Figure 3):

$$X \ := \ \{(sx - x) + (sy - y) : x \neq y \in \Psi\} \ \subseteq \ \Gamma.$$

In the standard model, the set X is countable and has the following explicit description:

$$X \ = \ \{(\underbrace{0, \ldots, 0}_{m}, 1, \underbrace{0, \ldots, 0}_{n}, 1, 0, 0, \ldots) : m \geq 1, n \geq 0\}.$$

To see that X is ∞-small using Lemma 3.9, consider the affine map:

$$F\colon \Psi^4 \to \Gamma, \quad (x_0, x_1, x_2, x_3,) \ \mapsto \ (x_0 - x_1) + (x_2 - x_3)$$

and note that $X \subseteq \mathrm{image}(F)$. Moreover, for the set:

$$W \ := \ \{(x_0, x_1, x_2, x_3) : x_0 = sx_1, x_2 = sx_3, x_1 < x_3\} \ \subseteq \ \Psi^4,$$

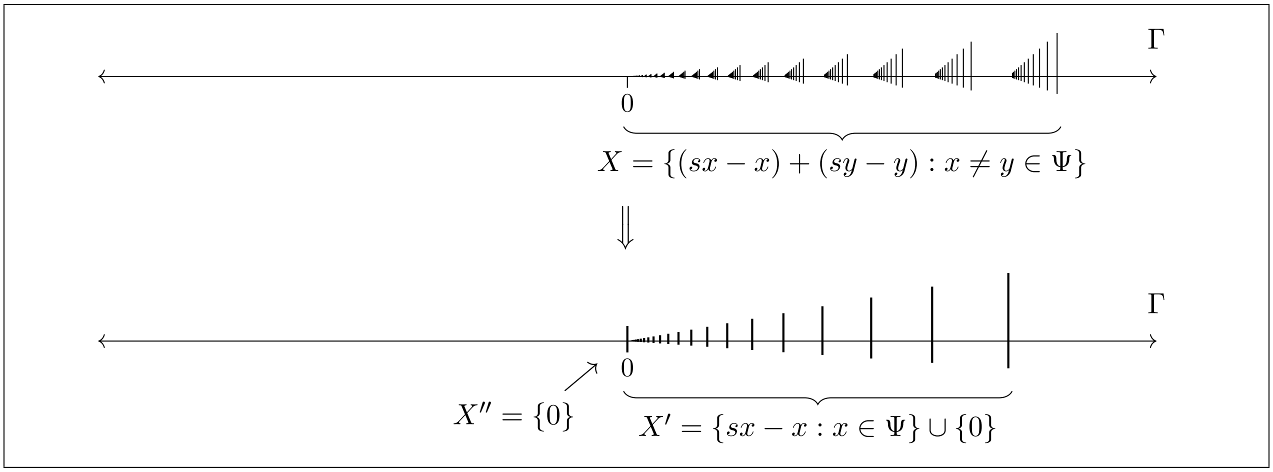


Figure 3: A set X with $X, X', X'' \neq \varnothing$ and $X^{(3)} = \varnothing$.

we have a bijection $F|_W \colon W \to X$, where W is definable in the structure $(\Psi; <)$. Next, observe that after taking the derived set finitely many times, we will arrive at $\varnothing$. Indeed:

$$\begin{aligned} X' &= \{sx - x : x \in \Psi\} \cup \{0\} \\ &= \{(\underbrace{0, \dots, 0}_{m}, 1, 0, 0, \dots) : m \geq 1\} \cup \{0\} \quad \text{(in the standard model)} \end{aligned}$$

and so $X'' = \{0\}$ and thus $X^{(3)} = \varnothing$. Next, consider the image $\pi_{s^n 0}(X)$ in the quotient $\Gamma/\Delta_{s^n 0}$, which we picture in Figure 4. Since in the standard model

$$\Delta_{s^n 0} = \{(\underbrace{0, \dots, 0}_{n}, r_n, r_{n+1}, \dots)\}$$

we have:

$$\Gamma/\Delta_{s^n 0} = \{(r_0, \dots, r_{n-1})\} \cong \mathbb{R}^n$$

and thus $\pi_{s^n 0}(X)$ is the set of 0/1-vectors of length n that begin with 0 and contain $\leq$ 2-many 1's. Observe that in this case $\pi_\phi(X)$ is finite, which indeed always happens when $\phi = s^n 0$ for some n (Corollary 3.26).

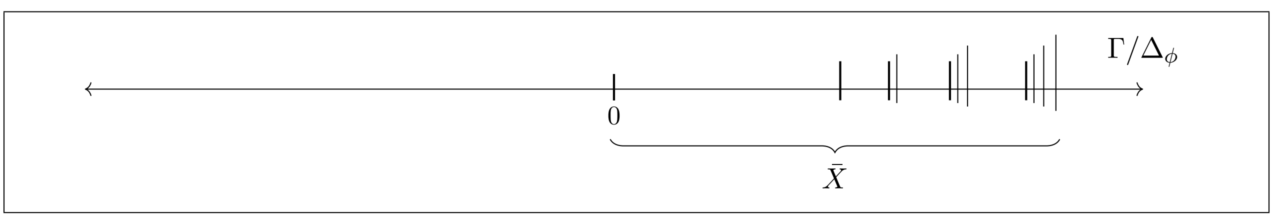


Figure 4: The image in the quotient Γ/Δ_ϕ of the set in Figure 3 for the value $\phi = s^5 0$.

Here is the general case.

Small Sets Theorem 3.11. *For $\phi \in \Psi_\infty$ and $X \subseteq \Gamma$ definable in (Γ, ψ), the following are equivalent:*

(1) $\dim_\phi X \leq 0$.
(2) $\bar{X}$ is a finite union of discrete sets.
(3) $\bar{X}$ has empty interior.
(4) X does not contain an interval (a, b) such that $b - a > \Delta_\phi$.
(5) X can be covered by finitely many affine maps $\Psi^n \times \Delta_\phi \to \Gamma$.

(6) $\bar{X}$ *is d-finite.*
(7) $\bar{X}$ *is nowhere dense*
(8) $\bar{X}$ *is definably meager in* $\bar{\Gamma}$*.*
(9) X *is* $\Psi\Delta_\phi$*-internal.*
(10) X *is* $\Psi\Delta_\phi$*-coanalysable.*

Furthermore, if (Γ, ψ) *is the standard model, then the above are additionally equivalent to:*

(11) $|\bar{X}| \leq \aleph_0$*.*
(12) $|\bar{X}| < 2^{\aleph_0}$*.*

If $\phi < \infty$*, then the above are additionally equivalent to:*

(13) $\bar{X}$ *is closed and discrete, and thus* $\bar{X}' = \varnothing$*.*

Finally, if $\phi = \infty$*, then the above are equivalent to:*

(14) X *is dp-finite.*
(15) The dp-rank of X *is less than* $\aleph_0$*.*

See Subsections 4.1, 4.5 and 4.6 for precise definitions of *definably meager*, *internality*, *coanalysability*, and *dp-rank*.

3.4. Proof of the Dimension Theorem from (1)⇔(4)⇔(5) of the Small Sets Theorem.

Corollary 3.12 (Uniqueness)**.** *Let* d *be a dimension function on* (Γ, ψ)*. Then* $\mathrm{d} = \dim_\phi$ *for some* $\phi \in \Psi_\infty$*.*

Proof. By Corollary 3.6, let ϕ be such that $\mathrm{d}\,\Delta_\phi = 0$ and $\mathrm{d}\,\Delta_\xi = 1$ for every $\xi \in \Psi^{<\phi}$. By Fact 3.2 it suffices to prove: for every definable $X \subseteq \Gamma_\infty$,

(a) if $\dim_\phi X \leq 0$, then $\mathrm{d}\,X \leq 0$, and
(b) if $\dim_\phi X = 1$, then $\mathrm{d}\,X = 1$.

Let $X \subseteq \Gamma_\infty$ be an arbitrary definable set. We may assume $X \subseteq \Gamma$ since removing ∞ does not change whether dimension is ≤ 0 or $= 1$. We have two cases:

- Suppose $\dim_\phi X \leq 0$. By (D2) and Lemma 3.9, we further reduce to the case $X = \mathrm{image}(F)$, where $F\colon \Psi^n \times \Delta_\phi \to \Gamma$ is an affine map. Since $\mathrm{d}\,\Psi = 0$ by Corollary 3.4, it follows from Fact 3.2(5a,5b) that $\mathrm{d}\,X \leq 0$.
- Suppose $\dim_\phi X = 1$. Then by (4)⇒(1), X contains an interval (a, b) where $b - a > \Delta_\phi$. Thus $\mathrm{d}(a, b) = 1$ by Corollary 3.6, and thus $\mathrm{d}\,X = 1$ by Fact 3.2(1). □

Corollary 3.13 (Existence)**.** *For each* $\phi \in \Psi_\infty$*, the function* $\dim_\phi$ *is a dimension function.*

Proof. It remains to show that each $\dim_\phi$ satisfies (D4). Suppose $S \subseteq \Gamma_\infty^{n+1}$ is definable and set $B_i \coloneqq \{a \in \Gamma_\infty^n : \dim_\phi S_a = i\}$ for $i = 0, 1$. Then using (1)⇔(4) we see that B_1 is definable:

$$\begin{aligned} B_1 &= \{a \in \Gamma_\infty^n : \dim_\phi(S_a \setminus \{\infty\}) = 1\} \\ &= \{a \in \Gamma_\infty^n : \text{there exists an interval } (a, b) \subseteq S_a \setminus \{\infty\} \text{ such that } b - a > \Delta_\phi\}. \end{aligned}$$

Thus B_0 is definable as well. It now follows from standard pregeometry arguments that $\dim_\phi S_i = \dim_\phi B_i + i$, where $S_i \coloneqq \{(x, y) \in S : x \in B_i\}$. This is asserted in [1, Proposition 2.7(2)]. □

Corollary 3.14. *Fix* $d \in \{-\infty, 0, \ldots, n\}$ *and suppose* $(I; \leq)$ *is a definable directed set and* $X \subseteq I \times \Gamma_\infty^n$ *is a definable family such that* $X_a \subseteq X_b$ *for every* $a \leq b$ *in* I*. If* $\dim_\phi X_a \leq d$ *for every* $a \in I$*, then* $\dim_\phi(\bigcup_{a \in I} X_a) \leq d$*.*

Proof. Let (Γ^*, ψ^*) be a monster model extending (Γ, ψ). Then Corollary 3.13 gives $\dim_\phi(X^*)_a \leq d$ for every $a \in I^*$. By Lemma 3.10 and [20, Lemma 3.71], it follows that $\dim_\phi(\bigcup_{a \in I^*} (X^*)_a) \leq d$. Thus $\dim_\phi(\bigcup_{a \in I} X_a) \leq d$ since $(\bigcup_{a \in I} X_a)^* = \bigcup_{a \in I^*} (X^*)_a$. □

3.5. **Higher arity definable sets.** For definable subsets of Γ^n with $n > 1$, the conditions in the Small Sets Theorem are clearly not equivalent. However, each turns out to be equivalent to either having ϕ-dimension ≤ 0 (being "very small") or having ϕ-dimension $< n$ (being "kinda small"). In this subsection, we freely use both the Small Sets Theorem and the Dimension Theorem.

Kinda Small Sets Theorem 3.15. *For $\phi \in \Psi_\infty$ and $X \subseteq \Gamma^n$ definable in (Γ, ψ) the following are equivalent:*

(1) $\dim_\phi X < n$.
(2) $\bar{X}$ has empty interior.
(3) X does not contain an open box $\prod_{i=1}^n (a_i, b_i)$ where $b_i - a_i > \Delta_\phi$ for each i.
(4) X can be covered by finitely many sets of the form

$$\{\alpha \in \Gamma^n : q \cdot \alpha \in Y\}$$

where $q \in \mathbb{Q}^n$ is not the zero tuple and $Y \subseteq \Gamma$ is a definable ϕ-small set
(5) $\bar{X}$ is nowhere dense.
(6) $\bar{X}$ is definably meager in $\bar{\Gamma}^n$.

Proof. Suppose $\dim_\phi X < n$, and let α be a tuple in an elementary extension (Γ^*, ψ^*) of (Γ, ψ) with $\alpha \in X^*$. Then $\mathrm{rk}_\phi(\alpha|\Gamma) < n$, so α is $\mathbb{Q}$-linearly dependent over

$$\mathrm{cl}_\phi(\Gamma) \;=\; \Gamma + \mathrm{span}_{\mathbb{Q}}(\Psi^*) + \Delta^*_\phi.$$

By a standard compactness argument, X is covered by finitely many sets of the form

$$\{\alpha \in \Gamma^n : q \cdot \alpha \in Y\}$$

where $q \in \mathbb{Q}^n$ is not the zero tuple and where Y is the image of an affine map $\Psi^m \times \Delta_\phi \to \Gamma$. This gives (1)⇒(4).

We now show that (4)⇒(5). Since nowhere dense sets form an ideal, we may assume that X is of the form $\{\alpha \in \Gamma^n : q \cdot \alpha \in Y\}$. Then $\bar{X} \subseteq \{\bar{\alpha} \in \bar{\Gamma}^n : q \cdot \bar{\alpha} \in \bar{Y}\}$. We may further assume that $\bar{Y}$ is closed—when $\phi < \infty$, this holds by (13) of the Small Sets Theorem, and when $\phi = \infty$, then $\bar{Y} = Y$ is nowhere dense, so its closure is as well. Then $\bar{X}$ is closed, so we need only show that $\bar{X}$ has empty interior. Since the map $\bar{\alpha} \mapsto q \cdot \bar{\alpha} \colon \bar{\Gamma}^n \to \bar{\Gamma}$ is open, this follows from the fact that $\bar{Y}$ has empty interior.

The implications (5)⇒(2)⇒(3) are immediate. We establish the implication (3)⇒(1) in Proposition 4.2 below. Finally, having established (5)⇒(1), the implications (5)⇒(6)⇒(1) follow as in the $n = 1$ case done in Stage (IV) of the proof of the Small Sets Theorem below, also using Corollary 3.14. □

Using Fact 3.2(6) and the equivalence (1)⇔(2) of the Kinda Small Sets Theorem, we obtain:

Corollary 3.16 (Coincidence with topological dimension)**.** *The dimension function* $\dim_\phi$ *coincides with naive topological dimension in the quotient $\bar{\Gamma}$. That is, for $X \subseteq \Gamma^n$ and $d \leq n$, we have* $\dim_\phi X \geq d$ *if and only if $\pi(\bar{X})$ has nonempty interior for some coordinate projection $\pi \colon \bar{\Gamma}^n \to \bar{\Gamma}^d$.*

Corollary 3.17 (Local linearity)**.** *Suppose $f \colon \Gamma^n \to \Gamma_\infty$ is definable. Then there exists a definable dense open set $U \subseteq \Gamma^n$ such that f is locally affine at all $x \in U$. Moreover, there is a finite set $Q \in \mathbb{Q}^n$ that the "slopes" of the affine functions come from.*

Proof. As $T_{\log}$ has quantifier elimination and a universal axiomatization in the language $\mathcal{L}_{\log}$, definable functions are given piecewise by terms. Thus, we may assume that f is a term τ. We proceed by induction on complexity of terms, assuming that for any term σ less complex than τ, there is a definable dense open set U_σ on which σ is locally affine (with slopes coming from a finite subset $Q_\sigma \subseteq \mathbb{Q}^n$). If τ is a sum $\sigma_1 + \sigma_2$, then this still holds for τ on the dense open set $U_{\sigma_1} \cap U_{\sigma_2}$, and if $\tau = -\sigma$ or $\delta_n(\sigma)$ for some $n \geq 1$, then this still holds for τ on U_σ, so we may assume that τ is either of the form $\psi \circ \sigma$, $s \circ \sigma$, or $p \circ \sigma$.

Let $V \subseteq U_\sigma$ be the set of $x \in \Gamma^n$ at which σ is locally constant, so τ is locally constant on V as well. Let $W \coloneqq U_\sigma \setminus V$, so σ is locally affine and nonconstant on W. We will find a dense open subset of W on which τ is locally constant. By Lemma 2.8, s is locally constant on Γ, ψ is locally constant on $\Gamma \setminus \{0\}$, and p is locally constant (with constant value ∞) on $\Gamma \setminus \Psi^{>s0}$. Thus, it is enough to show that

$$X \ \coloneqq\ \{x \in W : \sigma(x) \in \{0\} \cup \Psi^{>s0}\}$$

is nowhere dense.

For $q \in Q_\sigma$ and $\beta \in \Gamma$, let $W_{q,\beta}$ be the set of $x \in W$ such that $\sigma(y) = q \cdot y + \beta$ for y in an open neighbourhood of x. Then $(W_{q,\beta})_{q \in Q_\sigma, \beta \in \Gamma}$ is a definable family of disjoint open subsets of W with $W = \bigcup_{q,\beta} W_{q,\beta}$. Let

$$X_{q,\beta} \ \coloneqq\ \{x \in W_{q,\beta} : q \cdot x + \beta \in \{0\} \cup \Psi^{>s0}\},$$

so $X = \bigcup_{q,\beta} X_{q,\beta}$ and each $X_{q,\beta}$ is nowhere dense by the equivalence (4)⇔(5) of the Kinda Small Sets Theorem. Since the sets $W_{q,\beta}$ are disjoint, we see that X is also nowhere dense, as desired. ☐

Corollary 3.18 (Uniform definability of dimension)**.** *Suppose $S \subseteq \Gamma^n$ and $f \colon S \to \Gamma^m$ are definable and fix $d \in \{-\infty, 0, \ldots, n\}$. Then*

$$\{(\phi, b) : \dim_\phi f^{-1}(b) = d\} \ \subseteq\ \Psi_\infty \times \Gamma^m$$

is definable.

Proof. For any ϕ, we have $\dim_\phi f^{-1}(b) = -\infty$ if and only if $b \notin f(S)$, so we may assume $d \geq 0$. It suffices to show that $\{(\phi, b) : \dim_\phi f^{-1}(b) \geq d\}$ is definable. By Fact 3.2(6) and the equivalence (1)⇔(3) of the Kinda Small Sets Theorem, we have $\dim_\phi f^{-1}(b) \geq d$ if and only if $\pi(f^{-1}(b))$ contains an open box $\prod_{i=1}^d (a_i, b_i)$ with $b_i - a_i > \Delta_\phi$ for some coordinate projection $\pi \colon \Gamma^n \to \Gamma^d$. This is a definable condition on ϕ and b. ☐

Corollary 3.18 suggests considering the limits of $\dim_\phi$ and rk_ϕ as ϕ increases in Ψ Note that $\bigcap_{\phi \in \Psi} \Delta_\phi = \{0\} = \Delta_\infty$. This allows us to compute the ∞-dimension of a definable set X as a limit of its ϕ-dimensions.

Corollary 3.19 (Continuity of dimension)**.** *Suppose $X \subseteq \Gamma^n$ is definable. Then:*

$$\lim_{\phi \to \infty} \dim_\phi(X) \ =\ \dim_\infty(X).$$

Proof. Let $d \coloneqq \dim_\infty(X)$. Since $\dim_\phi(X)$ is increasing in ϕ, it is enough to show that $\dim_\phi(X) \geq d$ for some $\phi \in \Psi$. By Corollary 3.16, there is a coordinate projection $\pi \colon \Gamma^n \to \Gamma^d$ and an open box $\prod_{i=1}^d (a_i, b_i)$ contained in $\pi(X)$. Take ϕ large enough with $b_i - a_i > \Delta_\phi$ for $i = 1, \ldots, d$. For this ϕ, we have $\dim_\phi X \geq d$. ☐

This continuity of dimension can fail for the ranks of individual elements:

Example 3.20. Let (Γ, ψ) be the standard model, and let $\alpha = (1, 1/2, 1/3, 1/4, \ldots)$ be an element in an immediate elementary extension (Γ^*, ψ^*) of (Γ, ψ) (cf. [24, Example 2]). Then inside (Γ^*, ψ^*) we have $\alpha \notin \mathrm{cl}_\infty \Gamma$, although $\alpha \in \bigcap_{\phi \in \Psi} \mathrm{cl}_\phi \Gamma$. Thus, $\lim_{\phi \to \infty} \mathrm{rk}_\phi(\alpha|\Gamma) = 0 < 1 = \mathrm{rk}_\infty(\alpha|\Gamma)$.

The following says that the closure of a definable set $X \subseteq \Gamma^n$ in the initial topology induced by π_ϕ has the same ϕ-dimension as X. This confirms [20, Conjecture 9.11] in our setting.

Corollary 3.21. *Let $X \subseteq \Gamma^n$ be definable. Then $\dim_\phi \pi_\phi^{-1}(\mathrm{cl}_{\bar{\Gamma}^n}(\bar{X})) = \dim_\phi X$. Hence, X has the same ϕ-dimension as its topological closure in the topology coming from the order topology on Γ.*

Proof. From $X \subseteq \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X))$, we get $\dim_\phi X \le \dim_\phi \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X))$. Now observe that

$$\dim_\phi X \ < \ n \quad \Longleftrightarrow \quad \dim_\phi \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X)) \ < \ n,$$

which follows from (1)⇔(5) of the Kinda Small Sets Theorem and the fact that $\bar X$ is nowhere dense if and only if

$$\mathrm{cl}_{\bar\Gamma^n}(\bar X) \ = \ \pi_\phi(\pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X)))$$

is nowhere dense by Lemma A.3. Suppose $\dim_\phi \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X)) = d$ and use Fact 3.2(6) to take a projection $\pi\colon \Gamma^n \to \Gamma^d$ with $\dim_\phi \pi(\pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X))) = d$. Then

$$\pi(\pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^n}(\bar X))) \ = \ \pi_\phi^{-1}(\pi(\mathrm{cl}_{\bar\Gamma^n}(\bar X))) \ \subseteq \ \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^d}(\pi(\bar X))),$$

using Lemma A.1 for the containment. Hence $\dim_\phi \pi_\phi^{-1}(\mathrm{cl}_{\bar\Gamma^d}(\pi(\bar X))) = d$, so $\dim_\phi \pi(X) = d$ by the equivalence displayed above, yielding $\dim_\phi X = d$. □

Very Small Sets Theorem 3.22. *For $\phi \in \Psi_\infty$ and $X \subseteq \Gamma^n$ definable in (Γ, ψ) the following are equivalent:*

(1) $\dim_\phi X \le 0$.
(2) X is contained in a product $\prod_{i=1}^n X_i$, where each $X_i \subseteq \Gamma$ is a definable ϕ-small set.
(3) $\bar X$ is a finite union of discrete sets.
(4) X can be covered by finitely many affine maps $\Psi^m \times \Delta_\phi \to \Gamma^n$.
(5) $\bar X$ is d-finite.
(6) X is $\Psi\Delta_\phi$-internal.
(7) X is $\Psi\Delta_\phi$-coanalysable.

Furthermore, if (Γ, ψ) is the standard model, then the above are additionally equivalent to:

(8) $|\bar X| \le \aleph_0$.
(9) $|\bar X| < 2^{\aleph_0}$.

If $\phi < \infty$, then the above are additionally equivalent to:

(10) $\bar X$ is closed and discrete, and thus $\bar X' = \varnothing$.

Finally, if $\phi = \infty$, then the above are additionally equivalent to:

(11) X is dp-finite.
(12) The dp-rank of X is less than $\aleph_0$.

Proof. For (1)⇒(2), let X_i be the coordinate projection of X onto the ith coordinate. Then $X \subseteq \prod_{i=1}^n X_i$ and $\dim_\phi X_i \le \dim_\phi X \le 0$ by Fact 3.2(5b). Suppose (2) holds and for each $i \in \{1, \dots, n\}$, take affine maps $F_{i,0}, \dots, F_{i,n_i}\colon \Psi^m \times \Delta_\phi \to \Gamma$ whose images cover X_i (we arrange that m is the same for each $F_{i,j}$). For $\boldsymbol{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$ with $j_i \le n_i$ for each i, we let $F_{\boldsymbol{j}}\colon \Psi^m \times \Delta_\phi \to \Gamma^n$ be the affine map $(F_{1,j_1}, F_{2,j_2}, \dots, F_{n,j_n})$. Then X is covered by the images of the (finitely many) $F_{\boldsymbol{j}}$, establishing (4).

We have (4)⇒(6) by the definition of internality (see Lemma 4.16) and (6)⇒(7) since internality always implies coanalysability (see Remark 4.20). Using Lemma 4.21, we obtain (7)⇒(1), since $\dim_\phi \Psi\Delta_\phi = 0$.

Take $X_1, \dots, X_n \subseteq \Gamma$ as in (2), so the Small Sets Theorem gives natural numbers $m_1, \dots, m_n$ such that $\bar X_i^{(m_i)} = \varnothing$ for each i. Using Lemma A.17 and induction on n, we get $\bar X^{(m)} = \varnothing$ for $m = m_1 + \cdots + m_n$, so (2)⇒(5), and we always have (5)⇒(3) (see Lemma A.13).

Let $X \subseteq \Gamma^n$ and suppose that $\bar X$ is a finite union of discrete sets. To establish (3)⇒(1), we need to show that $\dim_\phi X \le 0$. We proceed by induction on n (the case $n = 1$ being the Small Sets

Theorem). Since $\bar{X}$ is nowhere dense (Lemma A.5), the Kinda Small Sets Theorem 3.15 tells us that X can be covered by finitely many sets of the form

$$\{\alpha \in \Gamma^n : q \cdot \alpha \in Y\}$$

where $q \in \mathbb{Q}^n$ is a tuple of rational numbers, not all zero, and $Y \subseteq \Gamma$ is definable and ϕ-small. Using (D2), we may assume that X is contained in one of these sets. For $\gamma \in Y$, let $X_\gamma := \{\alpha \in X : q \cdot \alpha = \gamma\}$, so X_γ is the intersection of X with the hyperplane $\{\alpha \in \Gamma^n : q \cdot \alpha = \gamma\}$, which is homeomorphic to Γ^{n-1} via some coordinate projection map $\pi\colon \Gamma^n \to \Gamma^{n-1}$. Clearly, $\overline{X_\gamma}$ is also a finite union of discrete sets, so $\pi(\overline{X_\gamma}) = \overline{\pi(X_\gamma)}$ is a finite union of discrete sets as well. Our induction hypothesis and Fact 3.2(5b) give $\dim_\phi X_\gamma = \dim_\phi \pi(X_\gamma) \leq 0$. Finally, using that $X = \bigcup_{\gamma \in Y} X_\gamma$, that $\dim_\phi Y \leq 0$, and Fact 3.2(5c), we get that $\dim_\phi X \leq 0$.

In the case that $\phi < \infty$, we obtain (2)⇒(10), using that a product of closed discrete sets is closed and discrete. Then (10)⇒(5) trivially. If $\phi = \infty$, then (2)⇒(11) by Fact 4.22, (11)⇒(12) is trivial, and (12)⇒(2) again by Fact 4.22, taking X_i to be the coordinate projection of X onto the ith coordinate. Finally, the equivalence (1)⇔(2) and basic properties of cardinality give (1)⇔(8)⇔(9) in the standard model. □

Remark 3.23. Let $X \subseteq \Gamma^n$ be definable with $\dim_\infty X \leq 0$. Then X is already definable in the reduct (Γ, Ψ) by (4) above, since the structure $(\Psi; <)$ is purely stably embedded.

Remark 3.24. The equivalence (8)⇔(9) above asserts that "the continuum hypothesis holds for definable sets in the standard model".

Corollary 3.25 (d-minimality)**.** *The theory $T_{\log}$ is d-minimal in the stronger sense defined by Fornasiero in* [20, Definition 9.1]*, i.e.:*

(1) If $X \subseteq \Gamma$ is definable with empty interior, then X is a finite union of discrete sets.
(2) If $X \subseteq \Gamma^n$ is definable and discrete, then $\pi_1(X)$ has empty interior, where π_1 is the projection onto the first coordinate.
(3) If $X \subseteq \Gamma^2$ and $U \subseteq \pi_1(X)$ are definable, U is open and nonempty, and X_a has nonempty interior for each $a \in U$, then X has nonempty interior.

Proof. The first condition holds by the Small Sets Theorem. For X as in the second condition, the Very Small Sets Theorem 3.22 gives $\dim_\infty \pi_1 X \leq \dim_\infty X \leq 0$, so $\pi_1 X$ has empty interior by the Small Sets Theorem. For X, U as in the third condition, we have $\dim_\infty U = 1 = \dim_\infty X_a$ for all $a \in U$ by the Small Sets Theorem. Then $\dim_\infty X = 2$, so X has nonempty interior by the Kinda Small Sets Theorem 3.15. □

Corollary 3.26. *Let $X \subseteq \Gamma^n$ be definable and suppose $\phi = s^k 0$ for some k. Then $\dim_\phi X \leq 0$ if and only if $\bar{X}$ is finite.*

Proof. The implication (10)⇒(1) of the Very Small Sets Theorem tells us that $\dim_\phi X \leq 0$ if $\bar{X}$ is finite. For the other implication, it suffices by (1)⇒(4) of the Very Small Sets Theorem to consider the case $X = \text{image}(F)$, where $F\colon \Psi^m \times \Delta_\phi \to \Gamma^n$ is an affine map. Define $G\colon (\Psi^{\leq \phi})^m \to \Gamma^n$ by $G(\alpha) := F(\alpha, 0)$. Then $\text{image}(G)$ is finite, and $\overline{\text{image}(G)} = \bar{X}$. □

Remark 3.27. The statements on local linearity (Corollary 3.17) and d-minimality (Corollary 3.25) are given only in the $\phi = \infty$ case for simplicity, although there exist uniform versions of these statements as well. For this one must work either on the quotients $\bar{\Gamma}^n$, or consider the initial topology on Γ^n induced by the projection $\pi_\phi\colon \Gamma^n \to \bar{\Gamma}^n$.

3.6. **Failure of elimination of imaginaries.** First observe the following lemma:

Lemma 3.28. *Let $X \subseteq \Gamma$ be a $\varnothing$-definable set and let E be a $\varnothing$-definable equivalence relation on X. Suppose that in the standard model E has uncountably many uncountable equivalence classes. Then there is no $\varnothing$-definable function $f\colon X \to \Gamma_\infty^n$ such that $f(x) = f(y) \Leftrightarrow xEy$ for $x, y \in X$.*

Proof. Suppose towards contradiction that there is such a function f. Let

$$Z \;=\; \{z \in f(X) : \dim_\infty f^{-1}(z) = 1\}.$$

By our assumption on E and the equivalence (1)⇔(11) in the Small Sets Theorem, $\dim_\infty(Z) \geq 1$. But then $\dim_\infty(X) \geq 2$ by Fact 3.2(5c), contradicting that $X \subseteq \Gamma$. □

This lemma can be used to show that $T_{\log}$ does not have elimination of imaginaries in the 1-sorted language $\mathcal{L}_{\log}$; see [39, Lemma 8.4.7]. Specifically, we consider the analogue of the so-called *RV sort*, which corresponds to the equivalence relation $\sim_\psi$ on $\Gamma^{\neq}$ defined via:

$$x \;\sim_\psi\; y \quad\Longleftrightarrow\quad \psi(x) \;<\; \psi(x-y).$$

Corollary 3.29. *The imaginary $\Gamma^{\neq}/{\sim_\psi}$ is not eliminated.*

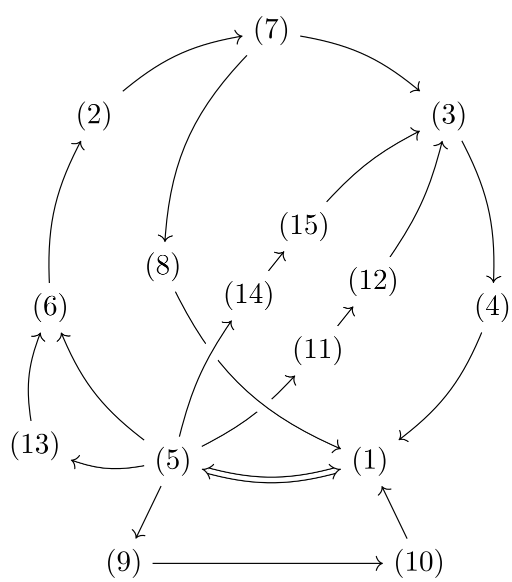
4. Proof of the Small Sets Theorem

In this section we prove the Small Sets Theorem as explained in Subsection 4.1 and indicated in the diagram below. To avoid any confusion, we emphasize that the Small Sets Theorem and the Dimension Theorem are proved simultaneously in three steps:

(1st) the equivalences between (1)-(7) and (13) of the Small Sets Theorem are established,
(2nd) the Dimension Theorem is then proved as explained in Subsection 3.4, and
(3rd) the remaining equivalences of the Small Sets Theorem are then established, *possibly with the help of the Dimension Theorem.*

In this section, we adopt the same global conventions as in Section 3.

4.1. **Roadmap of the proof.** We provide the proof in stages, with each stage increasing the set of properties proved to be equivalent to (1).



Stage (I): Incorporating (5). We start with (1)⇔(5), which is just Lemma 3.9.
Stage (II): Incorporating (2), (3), (4), (6), (7), *and* (13) *(Subsections 4.2, 4.3, and 4.4).*

- (4)⇒(1) Consider the contrapositive: suppose $\dim_\phi X = 1$. Then there is some α in an elementary extension of (Γ, ψ) such that $\alpha \in X^*$ and $\mathrm{rk}_\phi(\alpha|\Gamma) = 1$. Since $X \in \mathrm{tp}(\alpha|\Gamma)$, it follows from Proposition 4.2 that X contains an interval (a, b) such that $b - a > \Delta_\phi$.
- (5)⇒(6) (case $\phi = \infty$) This is Corollary 4.9, which follows from explicitly computing the derived set of the image of an affine map $\Psi^n \to \Gamma$ in Proposition 4.7. To put this in more topological terms: Lemma 4.8 shows that the image of an affine map $\Psi^n \to \Gamma$ is d-finite, so (6) follows from the fact that the d-finite sets form an ideal (see Lemma A.12).
- (5)⇒(13)⇒(6) (case $\phi < \infty$) The step (5)⇒(13) is Corollary 4.15. The direction (13)⇒(6) is trivial.
- (6)⇒(2) This direction is always true for arbitrary subsets of arbitrary topological spaces, i.e., the ideal of d-finite sets is always contained in the ideal generated by the discrete sets (see Lemma A.13). Note that in complete generality, the converse can fail (see Examples A.14 and A.15).
- (2) ⇒ (7) This direction is true for arbitrary subsets of arbitrary topological spaces that are T_1 and have no isolated points, i.e., the ideal generated by discrete sets is always contained in the ideal of nowhere dense sets under these assumptions; see Lemma A.5. Note that the converse will fail for definable subsets of Γ^n for $n > 1$.
- (7) ⇒ (3) This direction is always true; see Lemma A.4.
- (3)⇒(4) This step is trivial.

Note: At this point in the proof of the Small Sets Theorem we can prove the Dimension Theorem (Corollaries 3.12 and 3.13), and thus we may freely use the fact that each $\dim_\phi$ is a dimension function in the rest of the proof of the Small Sets Theorem.

Stage (III): Incorporating (8) *(definably meager).* We now say what it means for an arbitrary subset of $\bar{\Gamma}^n$ to be *definably meager*; recall that we are working in a 1-sorted setting, so we give a definition purely in terms of definable subsets of Γ^n.

Definition 4.1. Suppose $Z \subseteq \bar{\Gamma}^n$. We say that Z is **definably meager** in $\bar{\Gamma}^n$ if there exist a definable directed set $(I; \leq)$ and a definable family $Y \subseteq I \times \Gamma^n$ such that for every $a, b \in I$:

(1) $\overline{Y_a}$ is nowhere dense in $\bar{\Gamma}^n$,
(2) if $a \leq b$, then $\overline{Y_a} \subseteq \overline{Y_b}$, and
(3) $Z \subseteq \bigcup_a \overline{Y_a}$.

In this case, we say that $(I; \leq)$ and Y **witness** that Z is definably meager.

- (7)⇒(8) This direction is trivial. Indeed, if $\bar{X}$ is nowhere dense, then the singleton directed set $(\{*\}; \leq)$ and the definable family $Y \coloneqq \{*\} \times X$ witnesses that $\bar{X}$ is definably meager.
- (8)⇒(1) Suppose $\bar{X}$ is definably meager, witnessed by $(I; \leq)$ and $Y \subseteq I \times \Gamma$. In particular, each Y_a is ϕ-small by (7)⇒(1). By replacing Y_a with $\bigcup_{b \leq a} Y_b$ (which does not change the set $\overline{Y_a}$, so Y_a is still ϕ-small), we may assume that $Y_a \subseteq Y_b$ for $a \leq b \in I$. It now follows from Corollary 3.14 that $W \coloneqq \bigcup_{a \in I} Y_a$ also satisfies $\dim_\phi W \leq 0$. Thus $\bar{W}$ is nowhere dense by (1)⇒(7), and so $\bar{X} \subseteq \bar{W}$ is also nowhere dense.

Stage (IV): Incorporating (9) *and* (10) *(internality and coanalysability; Subsection 4.5).*

- (5)⇒(9) This is clear since (5) is expressing a precise form of *internality*; see Lemma 4.16.
- (9)⇒(10) This is because internality always implies coanalysability; see Remark 4.20.
- (10)⇒(1) If X is $\Psi\Delta_\phi$-coanalysable, then since $\dim_\phi \Psi\Delta_\phi = 0$, it follows from Lemma 4.21 that $\dim_\phi X \leq 0$. Note that this step uses the fact that $\dim_\phi$ is a dimension function, which we established after Stage (II) above.

Stage (V): Incorporating (11) *and* (12) *(cardinality in the standard model).*

- (5)⇒(11) If X satisfies (5), then $X \subseteq Y + \Delta_\phi$ for some ∞-small definable set Y. Since Y is countable in the standard model and $\bar{X} \subseteq \bar{Y}$, it follows that $|\bar{X}| \leq \aleph_0$.

- (11)⇒(12) This is trivial.
- (12)⇒(3) This follows easily from the observation that in the linear order $\Gamma_{\log}/\Delta_\phi$, each interval has size continuum.

Stage (VI): Incorporating (14) *and* (15) *(dp-rank; Subsection 4.6).*

- (5)⇒(14) This is by Fact 4.22, which uses that $\mathrm{dp}(\Psi) = 1$.
- (14)⇒(15) This is trivial.
- (15)⇒(3) This is Lemma 4.23.

4.2. **Finding open boxes.** Below, we show that any definable subset of Γ of ϕ-dimension 1 contains an open interval of width $> \Delta_\phi$, yielding (4)⇒(1) of the Small Sets Theorem. This is the unary case of the more general Proposition 4.2, which we need to establish the analogous direction (3)⇒(1) of the Kinda Small Sets Theorem. It is not clear to us whether this n-ary version can be deduced from the unary version.

Proposition 4.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a tuple in an elementary extension of (Γ, ψ) with $\mathrm{rk}_\phi(\alpha|\Gamma) = n$. Then every set X in $\mathrm{tp}(\alpha|\Gamma)$ contains an open box $\prod_{i=1}^n (a_i, b_i)$ where $b_i - a_i > \Delta_\phi$ for each i.*

Proof. We begin by replacing Γ with $\mathrm{cl}_\phi(\Gamma) \subseteq \Gamma\langle\alpha\rangle$; then Γ is still a model of $T_{\log}$ by Lemma 2.5 and we still have $\mathrm{rk}_\phi(\alpha|\Gamma) = n$, but we've arranged that $\Psi_{\Gamma\langle\alpha\rangle} = \Psi$, so

$$\Gamma\langle\alpha\rangle \;\cong\; \Gamma \oplus \bigoplus_{i=1}^n \mathbb{Q}\alpha_i.$$

Take δ in a further elementary extension (Γ^*, ψ^*) of $\Gamma\langle\alpha\rangle$ with

$$\Delta_\phi^* \;<\; \delta \;<\; \Gamma\langle\alpha\rangle^{>\Delta_\phi^*}.$$

Note that if $\phi < \infty$, then $\psi^*(\delta) = \phi$ and that if $\phi = \infty$, then $\psi^*(\delta) > \Psi$. By compactness, it is enough to show that if X is in $\mathrm{tp}(\alpha|\Gamma)$, then X^* contains the box $\prod_{i=1}^n (\alpha_i - \delta, \alpha_i + \delta)$. That is, we must show that for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (\Gamma^*)^n$ with $|\varepsilon_i| < \delta$ for each i, we have $\mathrm{tp}(\alpha|\Gamma) = \mathrm{tp}(\alpha + \varepsilon|\Gamma)$. Fix such an ε; then quantifier elimination allows us to further reduce the problem to finding an $\mathcal{L}_{\log}$-isomorphism $\iota\colon \Gamma\langle\alpha\rangle \to \Gamma\langle\alpha + \varepsilon\rangle$ over Γ with $\iota(\alpha) = \alpha + \varepsilon$.

Note that for any $\beta \in \Gamma\langle\alpha\rangle \setminus \Gamma$, we have $\psi^*(\beta) \leq \phi$. Thus, $|\beta| \in \Gamma\langle\alpha\rangle^{>\Delta_\phi^*}$, so $[\varepsilon_i] \leq [\delta] < [\beta]$. It follows that for $\gamma \in \Gamma$ and $q \in \mathbb{Q}^n$, the element $\gamma + q \cdot \alpha$ is positive if and only if $\gamma + q \cdot (\alpha + \varepsilon)$ is positive, so we have an ordered abelian group isomorphism

$$\iota\colon \Gamma \oplus \bigoplus_{i=1}^n \mathbb{Q}\alpha_i \to \Gamma \oplus \bigoplus_{i=1}^n \mathbb{Q}(\alpha_i + \varepsilon_i)$$

over Γ with $\iota(\alpha) = \alpha + \varepsilon$.

It remains to prove that ι commutes with ψ, s, and p. Consider an arbitrary element $\beta \in \Gamma\langle\alpha\rangle$; we can assume that $\beta \notin \Gamma$. Since $\Psi_{\Gamma\langle\alpha\rangle} = \Psi$, it is enough to show that $\psi^*(\iota(\beta)) = \psi^*(\beta)$, and likewise for s and p. Take $q \in \mathbb{Q}^n$ distinct from the zero tuple with $\iota(\beta) = \beta + q \cdot \varepsilon$. Since $[\varepsilon_i] < [\beta]$ for each i, we have $[\beta + q \cdot \varepsilon] = [\beta]$; in particular, $\psi^*(\beta + q \cdot \varepsilon) = \psi^*(\beta)$. The Integral Identity (Fact 2.2) gives $\int\beta = \beta - s\beta \in \Gamma\langle\alpha\rangle \setminus \Gamma$, so we have

$$(\textstyle\int\beta + q \cdot \varepsilon)' \;=\; \textstyle\int\beta + q \cdot \varepsilon + \psi^*(\int\beta + q \cdot \varepsilon) \;=\; \int\beta + q \cdot \varepsilon + \psi^*(\int\beta) \;=\; \beta + q \cdot \varepsilon.$$

It follows that $\int\beta + q \cdot \varepsilon = \int(\beta + q \cdot \varepsilon)$, and we again use the Integral Identity to conclude that $s\beta = \beta - \int\beta = \beta + q \cdot \varepsilon - \int(\beta + q \cdot \varepsilon) = s(\beta + q \cdot \varepsilon)$. Finally for p, we note that since $\beta \notin \Gamma \supseteq \Psi = \psi^*((\Gamma \oplus \mathbb{Q}\alpha)^{\neq})$ and the same for $\beta + q \cdot \varepsilon$, we have $p(\beta) = \infty = p(\beta + q \cdot \varepsilon)$. □

4.3. **Derived sets and Ψ-functions.** We establish here the direction (5)⇒(6) of the Small Sets Theorem in the case $\phi = \infty$.

For $\alpha, \beta \in \Gamma$ with $\beta \neq 0$, set $\alpha \prec_\psi \beta$ if $\psi(\alpha) > \psi(\beta)$, so if $\alpha \prec_\psi \beta$, then $[\alpha] < [\beta]$.

Lemma 4.3. *For every $\varepsilon \in \Gamma^>$, there exist $\delta_0, \delta_1 \in \Psi^{\geq \psi(\varepsilon)}$ such that $0 < \delta_0 - \delta_1 \prec_\psi \varepsilon$.*

Proof. Set $\delta_0 := s\psi(\varepsilon)$ and $\delta_1 := \psi(\varepsilon)$. Then $\delta_0 - \delta_1 = s\psi(\varepsilon) - \psi(\varepsilon) = -\textstyle\int\psi(\varepsilon)$ by the Integral Identity (Fact 2.2). It remains to note that $0 < -\textstyle\int\psi(\varepsilon) \prec_\psi \varepsilon$ by [3, Lemma 9.2.18(iii)]. □

Definition. A **Ψ-function** is $F\colon \Psi^I \to \Gamma$ defined by $F\alpha = \sum_{i \in I} q_i\alpha_i + \beta$ for $\alpha = (\alpha_i)_{i\in I} \in \Psi^I$, where $I \subseteq \mathbb{N}$ is finite, $(q_i)_{i\in I} \in (\mathbb{Q}^\times)^I$, and $\beta \in \Gamma$.

We allow $I = \varnothing$, in which case F has constant value β. Note that each ∞-small subset of Γ is contained in finite unions of images of Ψ-functions. Studying the limit points of such images will establish the part of the Small Sets Theorem claimed above.

In the remainder of this subsection, $F\colon \Psi^I \to \Gamma$ is a Ψ-function given by $F\alpha = \sum_{i\in I} q_i\alpha_i + \beta$ as in the definition above. We set $\|F\| := \sum_{i\in I} q_i$. For $J \subseteq I$, we define the Ψ-function $F_J\colon \Psi^J \to \Gamma$ by

$$F_J\alpha \ := \ \sum_{j\in J} q_j\alpha_j + \beta,$$

for $\alpha = (\alpha_j)_{j\in J} \in \Psi^J$, in which case $F\alpha = F_J(\alpha_j)_{j\in J} + F_{I\setminus J}(\alpha_i)_{i\in I\setminus J} - \beta$ for all $\alpha \in \Psi^I$. When convenient, we implicitly regard F as a function $\Psi^J \to \Gamma$, for $J \supseteq I$, by setting $q_j = 0$ for $j \in J \setminus I$. In particular, for $J \subseteq I$, we write $F_J\alpha$ instead of $F_J(\alpha_j)_{j\in J}$, for $\alpha \in \Psi^I$.

Recall that Ψ is $\mathbb{Q}$-linearly independent [23, Lemma 6.8], which yields the following uniqueness property.

Lemma 4.4. *If $G\colon \Psi^J \to \Gamma$ is a Ψ-function with $G\alpha = \sum_{j\in J} \tilde{q}_j\alpha_j + \tilde{\beta}$ such that $F\alpha = G\alpha$ for all $\alpha \in \Psi^{I\cup J}$, then $I = J$, $q_i = \tilde{q}_i$ for all $i \in I$, and $\beta = \tilde{\beta}$.*

Lemma 4.5. *Suppose that $\|F\| = 0$ and $I \neq \varnothing$. Then for every $\varepsilon \in \Gamma^>$ there exists $\alpha \in \Psi^I$ such that $0 < |F\alpha - \beta| \prec_\psi \varepsilon$.*

Proof. Pick $i_0 \in I$ and set $I_0 := I \setminus \{i_0\}$. Let $\varepsilon > 0$, so Lemma 4.3 gives $\delta_0, \delta_1 \in \Psi$ such that $0 < \delta_0 - \delta_1 \prec_\psi \varepsilon$. Then $\alpha \in \Psi^I$ defined by $\alpha_{i_0} := \delta_0$ and $\alpha_i := \delta_1$ for $i \in I_0$ satisfies

$$F\alpha - \beta \ = \ q_{i_0}\alpha_{i_0} + \textstyle\sum_{i\in I_0} q_i\alpha_i \ = \ q_{i_0}(\delta_0 + q_{i_0}^{-1}\sum_{i\in I_0} q_i\delta_1) \ = \ q_{i_0}(\delta_0 - \delta_1),$$

and thus $0 < |F\alpha - \beta| \prec_\psi \varepsilon$. □

In fact, the assumptions on F in the lemma above characterize when β is a limit point of image(F) in Γ.

Corollary 4.6. *We have $\beta \in \operatorname{image}(F)' \Leftrightarrow \|F\| = 0$ and $I \neq \varnothing$.*

Proof. If $\|F\| = 0$ and $I \neq \varnothing$, then Lemma 4.5 shows that $\beta \in \operatorname{image}(F)'$. Conversely, if $I = \varnothing$, then $\operatorname{image}(F) = \{\beta\}$ has no limit points, and if $\|F\| \neq 0$, then [23, Lemma 6.4] gives $\psi(F\alpha - \beta) = s0$ for all $\alpha \in \Psi^I$. □

Proposition 4.7. *We have*

$$\operatorname{image}(F)' \ = \ \bigcup_{\substack{\varnothing \neq J \subseteq I \\ \|F_J\| = 0}} \operatorname{image}(F_{I\setminus J}).$$

Proof. First, suppose that $\varnothing \neq J \subseteq I$ and $\|F_J\| = 0$. Let $(\alpha_i)_{i\in I\setminus J} \in \Psi^{I\setminus J}$ and $\varepsilon \in \Gamma^>$ be arbitrary. Lemma 4.5 gives $(\alpha_j)_{j\in J} \in \Psi^J$ so that $0 < |F_J\alpha - \beta| \prec_\psi \varepsilon$, where $\alpha \coloneqq (\alpha_i)_{i\in I}$. Note that

$$F_J\alpha \;=\; F_J\alpha + F_{I\setminus J}\alpha - F_{I\setminus J}\alpha \;=\; F\alpha + \beta - F_{I\setminus J}\alpha,$$

so $0 < |F_{I\setminus J}\alpha - F\alpha| \prec_\psi \varepsilon$. Hence $F_{I\setminus J}\alpha \in \operatorname{image}(F)'$.

For the reverse inclusion, take an elementary extension (Γ^*, ψ^*) of (Γ, ψ) containing an element $\delta \in (\Psi^*)^{>\Psi}$. By QE and UA, we arrange that $\Gamma^* = \Gamma\langle\delta\rangle$. The proof of [23, Lemma 4.11] shows

$$\Gamma\langle\delta\rangle \;=\; \Gamma \oplus \bigoplus_{k\in\mathbb{Z}} \mathbb{Q}s^k\delta$$

as an internal direct sum of $\mathbb{Q}$-linear subspaces, where $s^k\delta \coloneqq p^{-k}\delta$ for $k < 0$ in $\mathbb{Z}$. Fix $\gamma \in \Gamma$ and assume that for every $\varepsilon \in \Gamma^>$, there exists $\alpha \in \Psi^I$ such that $0 < |F\alpha - \gamma| < \varepsilon$. Set $\varepsilon(\delta) \coloneqq -\textstyle\int\delta$, which satisfies $0 < \varepsilon(\delta) < \Gamma^>$ since $\textstyle\int\Psi$ is cofinal in $\Gamma^<$ by [3, Lemma 9.2.15]. By elementarity, this yields $\alpha^* \in (\Psi^*)^I$ such that

$$0 \;<\; |F\alpha^* - \gamma| \;<\; \varepsilon(\delta).$$

It follows that $F\alpha^* \notin \Gamma$, so $J \coloneqq \{i \in I : \alpha_i^* > \Psi\} \neq \varnothing$. Note that $I \setminus J = \{i \in I : \alpha_i^* \in \Psi\}$. This gives for each $j \in J$ a $k_j \in \mathbb{Z}$ such that $\alpha_j^* = s^{k_j}\delta$. Setting $\tilde{\beta} \coloneqq \gamma - F_{I\setminus J}\alpha^* \in \Gamma$, we have

$$0 \;<\; \Big|\sum_{j\in J} q_j s^{k_j}\delta - \tilde{\beta}\Big| \;<\; \varepsilon(\delta).$$

By combining some of the coefficients $q_j \in \mathbb{Q}^\times$ and shrinking I if necessary, we arrange that the integers k_j are distinct. Note that we still have $J \neq \varnothing$. Then

$$\Psi^* \;\ni\; \psi\Big(\sum_{j\in J} q_j s^{k_j}\delta - \tilde{\beta}\Big) \;\geq\; \psi(\varepsilon(\delta)) \;=\; s\delta \;>\; \Psi.$$

Now, [23, Theorem 6.6] completely characterizes the values of $\psi(\sum_{j\in J} q_j s^{k_j}\delta - \tilde{\beta})$. Since $\psi(\tilde{\beta}) \in \Psi_\infty$, this characterization forces either $s^{k+1}\delta < \psi(\tilde{\beta})$ or $s^{k+1}\delta < s(q^{-1}\tilde{\beta})$, where $k \coloneqq \min_{j\in J}\{k_j\}$ and $q \coloneqq \sum_{j\in J} q_j$. The latter is impossible, since $s(q^{-1}\tilde{\beta}) \in \Psi < s^{k+1}\delta$. Hence $s^{k+1}\delta < \psi(\tilde{\beta})$, in which case $\Psi < s^{k+1}\delta$ gives $\psi(\tilde{\beta}) = \infty$. That is, $\tilde{\beta} = 0$, as desired. That $q = 0$ also follows from the case distinctions in [23, Theorem 6.6]. □

Lemma 4.8. *If* $|I| = n \geq 1$, *then* $\operatorname{image}(F)^{(n)} = \varnothing$.

Proof. Note that $\operatorname{image}(F)' = \{\beta\}' = \varnothing$ when $I = \varnothing$. By Proposition 4.7, we have

$$\operatorname{image}(F)' \;=\; \bigcup_{\substack{\varnothing\neq J\subseteq I\\ \|F_J\|=0}} \operatorname{image}(F_{I\setminus J}).$$

By induction on n, for each nonempty $J \subseteq I$ we have $\operatorname{image}(F_{I\setminus J})^{(n-1)} = \varnothing$. Then

$$\operatorname{image}(F)^{(n)} \;=\; \bigcup_{\substack{\varnothing\neq J\subseteq I\\ \|F_J\|=0}} \operatorname{image}(F_{I\setminus J})^{(n-1)} \;=\; \varnothing,$$

as desired (see Lemma A.9). □

Note that Lemma 4.8 only provides an upper bound. For example, the set $X = \Psi + \cdots + \Psi$ of n-fold sums satisfies $X' = \varnothing$.

The following gives (5)⇒(6) in the Small Sets Theorem in the case $\phi = \infty$:

Corollary 4.9. *If* A *is* ∞*-small, then* $A^{(n)} = \varnothing$ *for some* $n \in \mathbb{N}$.

Proof. This is immediate from Lemma 4.8 and the fact that d-finite sets form an ideal (see Lemma A.12). □

We have now completed enough of the Small Sets Theorem (i.e., Stage (II) in case $\phi = \infty$) to get that $\dim_\infty$ is a dimension function; see Corollary 3.13. This yields some additional uniformity in families, for which we define a **parametrized Ψ-function** to be $F\colon \Psi^n \times \Gamma \to \Gamma$ defined by

$$F(\alpha, \beta) \;=\; \sum_{i=0}^{n-1} q_i \alpha_i + \beta,$$

for $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \Psi^n$ and $\beta \in \Gamma$, where $q_i \in \mathbb{Q}^\times$ for $i = 0, \dots, n-1$.

Corollary 4.10. *If $X \subseteq \Gamma^{m+1}$ is definable, there exist parametrized Ψ-functions $F_i\colon \Psi^{n_i} \times \Gamma \to \Gamma$ for $i = 0, \dots, k-1$ such that for all $\delta \in \Gamma^m$ with $\dim_\infty X_\delta = 0$, we have*

$$X_\delta \;\subseteq\; \bigcup_{i=0}^{k-1} \operatorname{image}(F_i(\cdot, \beta_i))$$

for some $\beta_i \in \Gamma$.

Proof. This follows from [1, Lemma 2.7]. □

Corollary 4.11. *If $X \subseteq \Gamma^{m+1}$ is definable, then there is $N \in \mathbb{N}$ such that for all $\delta \in \Gamma^m$ with $\dim_\infty X_\delta = 0$, we have $X_\delta^{(N)} = \varnothing$.*

Proof. Note that $\operatorname{image}(F(\cdot, \beta))^{(n)} = \varnothing$ for all $\beta \in \Gamma$, where $F\colon \Psi^n \times \Gamma \to \Gamma$ is a parametrized Ψ-function, so this result follows from the previous corollary. □

4.4. **Equilateral sets and quotients.** In this subsection we establish (5)⇒(13) of the Small Sets Theorem, which follows from purely valuation-theoretic arguments. For that, let $(G, S; v)$ be an ordered abelian group with a convex (surjective) valuation $v\colon G \to S_\infty$, where S is an ordered set and we extend S to $S_\infty \coloneqq S \cup \{\infty\}$ in the usual way. Note that if S has no greatest element, then the valuation topology on G agrees with the order topology on G. On the other hand, if S has a maximum, then the valuation topology on G is discrete. See for instance [3, Chapter 2] for basic definitions and facts.

Let $s \in S$ and $X \subseteq G$. We say X is ***s*-equilateral** if $v(a - b) = s$ for all distinct $a, b \in X$. We say X is **anti-equilateral** if for every s, X contains no infinite s-equilateral subset. To see how this property passes to quotients, let $\Delta_s \coloneqq \{a \in G : va > s\}$, which is a convex subgroup of G. Then v induces a convex valuation $v\colon G/\Delta_s \to S_\infty^{\leq s}$ on the quotient ordered abelian group G/Δ_s, defined by $v(a + \Delta_s) = va$ for $a \in G \setminus \Delta_s$. Let $\bar{X}$ denote the image of X under the quotient map $G \to G/\Delta_s$.

Lemma 4.12. *If $X \subseteq G$ is anti-equilateral, then $\bar{X} \subseteq G/\Delta_s$ is anti-equilateral.*

Proof. Let $X \subseteq G$ and suppose that $\bar{X} \subseteq G/\Delta_s$ is an infinite s'-equilateral set, where $s' \in S^{\leq s}$. Take $Y \subseteq X$ so that the quotient map $G \to G/\Delta_s$ restricts to a bijection $Y \to \bar{X}$. Then Y is infinite and it is easy to check that Y is s'-equilateral, for if $y_1, y_2 \in Y$ with $y_1 \neq y_2$, then $y_1 - y_2 \notin \Delta_s$ and so

$$v(y_1 - y_2) \;=\; v\big((y_1 - y_2) + \Delta_s\big) \;=\; v\big((y_1 + \Delta_s) - (y_2 + \Delta_s)\big) \;=\; s'. \qquad \square$$

Lemma 4.13. *Suppose that S has a maximum $s = \max S$ and $X \subseteq G$ is anti-equilateral. Then X is closed and discrete in the order topology on G.*

Proof. We can assume that G is not discrete. To show that X is discrete, let $x \in X$. Then we have an interval (a, b) in G with $x \in (a, b)$ and $v(b - a) = s$. It follows that (a, b) is an infinite s-equilateral set, so $(a, b) \cap X$ is finite. By shrinking (a, b) further, we arrange that $(a, b) \cap X = \{x\}$, as desired. Similarly, if $x \in \mathrm{cl}(X)$, then $x \in X$. $\square$

The previous lemma only uses that X contains no infinite s-equilateral subset. Now we apply these lemmas to the setting $(\Gamma, \psi) \models T_{\log}$, construed as a structure $(\Gamma, \Psi; \psi)$ in the notation above.

Lemma 4.14. *Every ∞-small $X \subseteq \Gamma$ is anti-equilateral.*

Proof. Let $F\colon \Psi^n \to \Gamma$ be a Ψ-function and fix $\phi \in \Psi$. By the Pigeonhole Principle, it suffices to prove that $\mathrm{image}(F)$ contains no infinite ϕ-equilateral subset. Let $Y \subseteq \Psi^n$ be infinite and, for $k = 0, \dots, n-1$, let $\pi_k\colon Y \to \Psi$ be projection onto coordinate k. If $\pi_0^{-1}(p\phi)$ is finite, replacing Y by $Y \setminus \pi_0^{-1}(p\phi)$ arranges that $\pi_0^{-1}(p\phi) = \varnothing$ while keeping Y infinite. If $\pi_0^{-1}(p\phi)$ is infinite, replacing Y by $\pi_0^{-1}(p\phi)$ arranges that $\pi_0^{-1}(p\phi) = Y$ while keeping Y infinite. Repeating this procedure ensures that for each $k = 0, \dots, n-1$, either $\pi_k^{-1}(p\phi) = \varnothing$ or $\pi_k^{-1}(p\phi) = Y$. Let $I = \{k \in \{0, \dots, n-1\} : \pi_k^{-1}(p\phi) = \varnothing\}$, so $\psi(F(\alpha) - F(\beta)) = \psi(F_I(\alpha) - F_I(\beta))$ for all $\alpha, \beta \in Y$. For fixed $\alpha, \beta \in Y$, by combining the coefficients of the equal α_i and β_j, $i, j \in I$, we see that $F_I(\alpha) - F_I(\beta)$ is equal to a $\mathbb{Q}$-linear combination of distinct such α_i, β_j with coefficients summing to 0. Then by [23, Lemma 6.4], $\psi(F_I(\alpha) - F_I(\beta)) \in \{s\alpha_i, s\beta_i : i \in I\} \cup \{\infty\}$. In particular, $\psi(F(\alpha) - F(\beta)) \neq \phi$ for all $\alpha, \beta \in Y$. Now, if $Y \subseteq \Psi^n$ is such that $F(Y) \subseteq \mathrm{image}(F)$ is ϕ-equilateral, then the above shows that $F(Y)$ is finite. $\square$

Note that the quotient map $\pi_\phi\colon \Gamma \to \Gamma/\Delta_\phi$ is injective on $\Psi^{<s\phi}$ by the Successor Identity (Fact 2.2). In particular, the value set $\Psi^{<s\phi}$ of Γ/Δ_ϕ is order isomorphic to $\bar{\Psi}$ via π_ϕ, and we consider Γ/Δ_ϕ as a convexly valued ordered abelian group with $\psi\colon (\Gamma/\Delta_\phi)^{\neq} \to \bar{\Psi}$. Moreover, $(\Gamma/\Delta_\phi, \psi)$ is an asymptotic couple by [3, Lemma 9.2.24], but that is not needed here.

Corollary 4.15. *If $X \subseteq \Gamma$ is ϕ-small and $\phi \in \Psi$, then $\bar{X} \subseteq \Gamma/\Delta_\phi$ is closed and discrete.*

Proof. Let $Y \subseteq \Gamma$ be ∞-small, so Y is anti-equilateral. Note that $\psi((\Gamma/\Delta_\phi)^{\neq})$ has a maximum $\bar{\phi}$. Then by Lemmas 4.12 and 4.13, $\bar{Y}$ is closed and discrete in Γ/Δ_ϕ. It remains to note that for every ϕ-small $X \subseteq \Gamma$, there is an ∞-small $Y \subseteq \Gamma$ with $\bar{Y} = \bar{X}$. $\square$

4.5. **Internality and coanalysability.** We now investigate how previous properties are connected to the model-theoretic notion of *internality* to a definable set, namely internality to the distinguished family of sets $\Psi \cup \Delta_\phi$.

Definition. Let $X \subseteq \Gamma_\infty^n$ be definable. We say that X is **$\Psi\Delta_\phi$-internal** if $X \subseteq \mathrm{image}(f)$ for some definable $f\colon (\Psi \cup \Delta_\phi)^m \to \Gamma_\infty^n$.

Note that the collection of $\Psi\Delta_\phi$-internal definable subsets of Γ_∞^n (for a fixed n) form an ideal of definable sets on Γ_∞^n. Hence:

Lemma 4.16. *If $X \subseteq \Gamma^n$ can be covered by finitely many affine maps $\Psi^m \times \Delta_\phi \to \Gamma^n$, then X is $\Psi\Delta_\phi$-internal.*

In case $\phi = \infty$, [23, Theorem 6.3] shows that each definable $F\colon \Psi \to \Gamma_\infty$ is given piecewise by so-called s-functions. Now we generalize this notion in order to characterize definable functions $F\colon \Psi^n \to \Gamma_\infty$ in Proposition 4.17, answering a question Hieronymi asked the first-named author. This proposition is not used except in Corollary 4.19. In particular, our proof of the Small Sets Theorem does not require it. Nevertheless, it gives a more precise description of Ψ-internal sets.

Definition. We call $F\colon \Psi^m \to \Gamma_\infty$ a **generalized s-function** if for $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \in \Psi^m$

$$F(\alpha) \ = \ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} q_{i,j} s^{k_j}(\alpha_i) + \beta,$$

where $\beta \in \Gamma_\infty$, $q_{i,j} \in \mathbb{Q}$, and $k_j \in \mathbb{Z}$ for $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$. Here, $s^k \coloneqq p^{-k}$ for $k \in \mathbb{Z}^{<}$.

Proposition 4.17. *Let $F\colon \Psi^n \to \Gamma_\infty$ be definable. Then F is given piecewise by (finitely many) generalized s-functions.*

Proof. Let (Γ^*, ψ^*) be a $|\Gamma|^+$-saturated elementary extension of (Γ, ψ) and $(\alpha_0^*, \dots, \alpha_{n-1}^*) \in (\Psi^*)^n$. By universal axiomatization and quantifier elimination, we have $F(\alpha_0^*, \dots, \alpha_{n-1}^*) \in \Gamma\langle \alpha_0^*, \dots, \alpha_{n-1}^*\rangle$. Let $i_0 = \min\{i < n : \alpha_i^* \notin \Psi\}$. Then either $\alpha_{i_0}^* > \Psi$ or there is a nonempty downward closed $B \subsetneq \Psi$ with $s(B) \subseteq B$ and $B < \alpha_{i_0}^* < \Psi^{>B}$. In the former case appealing to [23, Lemma 4.11] and its proof, and in the latter to [23, Lemma 4.12] and its proof, we have $\Gamma\langle \alpha_{i_0}^*\rangle = \Gamma \oplus \bigoplus_{k\in\mathbb{Z}} \mathbb{Q} s^k(\alpha_{i_0}^*)$ as $\mathbb{Q}$-linear subspaces of Γ^*. Note that in applying [23, Lemma 4.12], condition 2. is satisfied because $B < \alpha_{i_0}^* < \Psi^{>B}$, $s(B) \subseteq B$, and s is increasing on the downward closure of Ψ in Γ by [23, Corollary 3.6]. Now take $i_1 = \min\{i < n : \alpha_i^* \notin \Psi \cup \{s^k(\alpha_{i_0}^*) : k \in \mathbb{Z}\}\}$ and proceed by induction to obtain $\{i_0, \dots, i_{m-1}\} \subseteq n$ with

$$\Gamma\langle \alpha_0^*, \dots, \alpha_{n-1}^*\rangle \ = \ \Gamma \oplus \bigoplus_{j=0}^{m-1} \bigoplus_{k\in\mathbb{Z}} \mathbb{Q} s^k(\alpha_{i_j}^*).$$

Then compactness yields a covering of $F\colon \Psi^n \to \Gamma_\infty$ by finitely many generalized s-functions. It follows that F is given piecewise by these generalized s-functions. $\square$

One could give a proof without using compactness by doing induction on n and terms and using [23, Theorem 6.6 and Corollary 6.7] in a fixed model of $T_{\log}$.

Lemma 4.18. *Let $F\colon \Psi^m \to \Gamma_\infty$ be a generalized s-function. Then* $\operatorname{image}(F) \subseteq \operatorname{image}(G) \cup \{\infty\}$ *for some Ψ-function $G\colon \Psi^I \to \Gamma$.*

Corollary 4.19. *The image of every definable $F\colon \Psi^n \to \Gamma$ is ∞-small.*

Note that this corollary gives a direct proof of (9)⇒(5) for $\phi = \infty$ in the Small Sets Theorem. Next we connect internality to the related model-theoretic notion of being coanalysable relative to a definable set, from [27]. We follow the presentation of [4]. As for internality, we focus on coanalysability relative to $\Psi\Delta_\phi$.

Definition. Let $X \subseteq \Gamma^n$ be defined by an $\mathcal{L}_{\log} \cup \{\phi, \gamma\}$-formula, where $\gamma \in \Gamma^m$. We say that X is **$\Psi\Delta_\phi$-coanalysable** if for all extensions $(\Gamma_1, \psi_1) \preccurlyeq (\Gamma_2, \psi_2) \models \operatorname{Th}_{\mathcal{L}_{\log}\cup\{\phi,\gamma\}}(\Gamma, \psi)$, if the Ψ-set and the subgroup Δ_ϕ do not grow, then neither does the interpretation of X.

This definition is the relevant specific instance of a general model-theoretic definition, and is justified by [27, Proposition 2.4], which provides equivalent formulations in that general context; see also [4, Section 6] for an exposition. What we need here is the following observation.

Remark 4.20. If $X \subseteq \Gamma_\infty^n$ is definable and $\Psi\Delta_\phi$-internal, then X is $\Psi\Delta_\phi$-coanalysable.

In fact, using the appropriate general definitions, internality always implies coanalysability, but the converse fails in some settings, including in the differential field $\mathbb{T}$ of logarithmic-exponential transseries, where internality and coanalysability are taken relative to the constant field [4]. Conversely, we show that $\Psi\Delta_\phi$-internality and $\Psi\Delta_\phi$-coanalzyability are equivalent in models of $T_{\log}$. For this, we connect $\Psi\Delta_\phi$-coanalysability to $\dim_\phi$.

Lemma 4.21. *If $X \subseteq \Gamma^n_\infty$ is definable and $\Psi\Delta_\phi$-coanalysable, then* $\dim_\phi X \leq 0$.

Proof. Since $\dim_\phi \Psi\Delta_\phi = 0$ and $\dim_\phi$ is a dimension function, this is just [1, Corollary 2.9]. □

4.6. **Dp-rank.** Let X be a definable set and κ a cardinal. Then the dp-rank of X is less than κ (written $\mathrm{dp}(X) < \kappa$) if, in every elementary extension (Γ^*, ψ^*) of (Γ, ψ), there does not exist a collection of formulas $(\phi_\alpha(x, y_\alpha))_{\alpha<\kappa}$, an array of elements $(b_{\alpha,i})_{\alpha<\kappa, i<\omega}$ from Γ^* and a family of elements $(a_\eta)_{\eta\in\omega^\kappa}$ from X^* such that

$$\phi_\alpha(a_\eta, b_{\alpha,j}) \quad \Longleftrightarrow \quad j = \eta(i) \quad \text{for all } \alpha, j.$$

The dp-rank of X is equal to κ (written $\mathrm{dp}(X) = \kappa$) if $\mathrm{dp}(X) < \kappa^+$ but $\mathrm{dp}(X) \not< \kappa$. We say that X is **dp-finite** if $\mathrm{dp}(X) = n$ for some n. Note that $\mathrm{dp}(X) < \aleph_0$ does not imply dp-finite in general, as $\mathrm{dp}(X)$ may be less than $\aleph_0$ but not less than any n.

Fact 4.22. *Let Y be another definable set and let f be a definable function.*

(1) If $X \subseteq Y$, then $\mathrm{dp}(X) \leq \mathrm{dp}(Y)$.
(2) If X, Y are dp-finite, then so is $X \times Y$.
(3) $\mathrm{dp}(f(X)) \leq \mathrm{dp}(X)$.
(4) $\mathrm{dp}(\Psi) = 1$.

Proof. (1) and (3) are straightforward and (2) holds by [30]. For (4), we know by Fact 2.6 that Ψ is stably embedded as a model of $\mathrm{Th}(\mathbb{N}, <)$, and this theory has dp-rank one [14]. □

Lemma 4.23. *Suppose that $X \subseteq \Gamma$ has nonempty interior. Then* $\mathrm{dp}(X) = \aleph_0$.

Proof. First, since $T_{\log}$ is countable and NIP, we have $\mathrm{dp}(X) < \aleph_1$; see [30, Remark 2.3]. Since X has nonempty interior, it contains a coset of Δ_ϕ for some sufficiently large $\phi \in \Psi$. By Fact 4.22, it is enough to show that $\mathrm{dp}(\Delta_\phi) = \aleph_0$. Arguing as in [26, Theorem 3.6 (8)], one can easily find for each $\varepsilon \in \Delta^>_\phi$ an infinite definable discrete subset of $(0, \varepsilon)$, so $\mathrm{dp}(X) \not< \aleph_0$ by [13, Theorem 2.11]. □

5. D-minimality criterion and applications

In this section, we give a general criterion for when the expansion of a topological theory by a collection of unary functions is d-minimal. This criterion systematizes a number of earlier results, dating back to van den Dries's 1985 proof that the real field with a predicate for the integer powers of two is d-minimal [15]. Instead of working with the predicate directly, van den Dries instead considers the real field expanded by the function λ that takes $x > 0$ to the largest power of two less than or equal to x. It is shown by a straightforward induction that terms in this extended language are given locally by semialgebraic functions, off of a finite union of discrete sets. D-minimality follows from this fact and a quantifier elimination result.

D-minimality for $(\mathbb{R}, 2^{\mathbb{Z}})$ was generalized by Miller, who showed that $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})$ is d-minimal for any polynomially bounded o-minimal expansion $\tilde{\mathbb{R}}$ of the real field with field of exponents $\mathbb{Q}$ and any $\alpha > 0$ [34]. The general structure of the proof is essentially the same, though some new facts about valuation theory for o-minimal structures are needed for the quantifier elimination result. This was further extended by Friedman and Miller, who showed that d-minimality for $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})$ is preserved after adding all subsets of all cartesian powers of $\alpha^{\mathbb{Z}}$ [22].

This process of describing terms in an expansion locally by functions definable in a base theory is relatively straightforward over the reals, but takes a bit more care when working over arbitrary topological structures. Here, we isolate sufficient conditions that allow us to prepare terms in this way. These "preparations," made explicit here, were used in [29] to show that the expansion of a power bounded o-minimal field $\boldsymbol{R}$ by a monomial group $\mathfrak{M}$ (that is, the image of a section of a T-convex valuation) is d-minimal.

Below is a list of structures and theories that can be easily shown to be d-minimal, using our criterion along with known quantifier elimination results. We note that our criterion does not (to our knowledge) establish d-minimality in the stronger sense defined by Fornasiero in [20, Definition 9.1]. In particular, we see no way to prove Corollary 3.25 without the Kinda Small and Very Small Sets Theorems.

(1) Our criterion generalizes the methods used to establish d-minimality for $(\mathbb{R}, 2^{\mathbb{Z}})$, $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})$, and $(\boldsymbol{R}, \mathfrak{M})$ as mentioned above, so these examples fall under our framework.
(2) The field $(\mathbb{Q}_p, p^{\mathbb{Z}})$ of p-adics with a predicate for the powers of p. This can be construed as a two-sorted structure $(\mathbb{Q}_p, \mathbb{Z}; v, \pi)$, where $v\colon \mathbb{Q}_p^\times \to \mathbb{Z}$ is the p-adic valuation and where $\pi\colon \mathbb{Z} \to \mathbb{Q}_p$ is the cross-section $z \mapsto p^z$. Then d-minimality follows from our criterion below, along with Ax and Kochen's quantifier elimination [6]. D-minimality was shown by Scowcroft, also using an induction on complexity of terms [37].
(3) The expansion of an o-minimal structure by an *iteration sequence* (a predicate for the iterates of a sufficiently fast-growing definable function). Quantifier elimination and d-minimality were established by Miller and Tyne [35].
(4) A *tame pair* of o-minimal fields: an o-minimal field $\boldsymbol{R}$ expanded by a predicate for a proper elementary substructure $\boldsymbol{A} \prec \boldsymbol{R}$ that is Dedekind complete in $\boldsymbol{R}$. Van den Dries and Lewenberg showed that these expansions eliminate quantifiers when considered with the standard part map $R \to A$, which takes x in the convex hull of A to the closest element in A. D-minimality follows from quantifier elimination and our criterion. To our knowledge, d-minimality has not been previously observed, though it follows for tame pairs of real closed ordered fields by d-minimality of the differential field $\mathbb{T}$ of logarithmic-exponential transseries, which is an expansion of such a pair; see [3, Corollary 16.6.11].
(5) D-minimality for $T_{\log}$ can quickly be established using our criterion and the quantifier elimination result in [23], as opposed to using the Small Sets Theorem. We describe how in Subsection 5.2.
(6) In Subsection 5.3, we establish d-minimality for henselian valued fields of residue characteristic zero, equipped with a section of the valuation and a lift of the residue field. This makes use of the quantifier elimination in [17].
(7) In Subsection 5.4, we show how one can use our criterion to show that $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})^\#$, the expansion of a polynomially bounded o-minimal expansion $\tilde{\mathbb{R}}$ of the real field with field of exponents $\mathbb{Q}$ by all subsets of all cartesian powers of $\alpha^{\mathbb{Z}}$ for some $\alpha > 0$, is d-minimal. As mentioned above, this was originally shown by Friedman and Miller, albeit by very different methods [22]. We approach this by showing that the quantifier elimination result of Miller for $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})$ can be extended to this expansion by considering this structure as a two-sorted structure.

5.1. **The d-minimality criterion.** For the remainder of this section, $\mathcal{L}$ is a multi-sorted language in the sorts $\mathcal{S}$, s ranges over S, $\boldsymbol{M}$ is an $\mathcal{L}$-structure, and T is a complete $\mathcal{L}$-theory.

We also fix a family $\chi = (\chi_s(x_s; y_s))_{s\in\mathcal{S}}$ of partitioned $\mathcal{L}$-formulas $\chi_s(x_s; y_s)$ where x_s is a variable in the sort s, and y_s is a multivariable.

Definition 5.1. We say that $\boldsymbol{M}$ is a **topological $\mathcal{L}$-structure** (with respect to χ) if for each $s \in \mathcal{S}$, the family:

$$\{\chi_s(M_s; a) : a \in M_{y_s}\}$$

is a basis for a topology on M_s. We say that T is a **topological $\mathcal{L}$-theory** (with respect to χ) if $\boldsymbol{M}$ is a topological $\mathcal{L}$-structure (with respect to χ) for every model $\boldsymbol{M}$ of T.

Note that if $\boldsymbol{M}$ is a topological $\mathcal{L}$-structure, then $\mathrm{Th}_{\mathcal{L}}(\boldsymbol{M})$ is a topological $\mathcal{L}$-theory. For a tuple of sorts $\boldsymbol{s}$, we let $M_{\boldsymbol{s}}$ denote the corresponding cartesian product. When $\boldsymbol{M}$ is a topological $\mathcal{L}$-structure, we construe such a cartesian product $M_{\boldsymbol{s}}$ as a topological space using the topology generated by the χ_s and the product topology. If $X \subseteq M_{\boldsymbol{s}}$ is definable, then $\mathrm{int}(X)$ and $\mathrm{br}(X)$ are also definable.

For the rest of this section $\boldsymbol{M}$ is a topological $\mathcal{L}$-structure and T is a topological $\mathcal{L}$-theory.

We say that $\boldsymbol{M}$ is T_1 at s if the topology on M_s is T_1, and we say that T is T_1 at s if every model $\boldsymbol{M}$ of T is T_1 at s. Note that if $\boldsymbol{M}$ is T_1 at s, then so is $\mathrm{Th}_{\mathcal{L}}(\boldsymbol{M})$.

Definition 5.2. Suppose $\boldsymbol{M}$ and T are T_1 at s. We say that $\boldsymbol{M}$ is **d-minimal** at s if every definable subset of M_s either has interior or is the union of finitely many discrete subsets of M_s. We say that T is **d-minimal** at s if $\boldsymbol{M}$ is d-minimal at s for every model $\boldsymbol{M}$ of T.

We fix a distinguished sort $s_0 \in \mathcal{S}$, and we write M_0 in place of M_{s_0}. We make the following assumptions on our topological $\mathcal{L}$-theory T:

(I) T is T_1 and d-minimal at s_0.
(II) For every model $\boldsymbol{M} \models T$, every $s \in \mathcal{S}$ and every $\mathcal{L}$-definable function $g\colon M_0 \to M_s$, the set $\mathrm{cl}(\mathsf{Discont}(g))$ is a finite union of discrete sets.

Remark 5.3. By assuming that T (and not just $\boldsymbol{M}$) is d-minimal at s_0, we obtain a uniform version of (II): For every finite tuple of sorts $\boldsymbol{s}$, every $s \in \mathcal{S}$, and every $\mathcal{L}$-definable function $g\colon M_{\boldsymbol{s}} \times M_0 \to M_s$, there is N such that $\mathrm{cl}(\mathsf{Discont}(g_x))$ is a union of N discrete sets for all $x \in M_{\boldsymbol{s}}$. This uses that being a union of at most N discrete sets is a definable condition; see Remark A.16.

Remark 5.4. Note that (II) does not follow from (I) in general, even though the set of discontinuities is definable. Indeed, Johnson notes that in the real field with the Sorgenfrey (lower limit) topology, every definable set with empty interior is finite, but $x \mapsto -x$ is nowhere continuous [28, Remark 1.16].

Consider $\mathcal{L}(\mathfrak{F}) \coloneqq \mathcal{L} \cup \mathfrak{F}$, where $\mathfrak{F}$ is a set of new unary function symbols. Let $T(\mathfrak{F})$ be a complete $\mathcal{L}(\mathfrak{F})$-theory extending T. Given $\boldsymbol{M} \models T$ we denote by $(\boldsymbol{M}, \mathfrak{F})$ an expansion of $\boldsymbol{M}$ to a model of $T(\mathfrak{F})$.

Proposition 5.5 (d-minimality criterion)**.** *Suppose the following conditions hold:*

(A) For every $\mathcal{L}(\mathfrak{F})$-formula $\varphi(t)$ where t is a unary variable in the sort s_0, there exist an $\mathcal{L}$-formula $\varphi^*(x_1, \ldots, x_n)$ and $\mathcal{L}(\mathfrak{F})$-terms $\tau_1(t), \ldots, \tau_n(t)$ such that:

$$T(\mathfrak{F}) \ \vdash \ \varphi(t) \leftrightarrow \varphi^*(\tau_1(t), \ldots, \tau_n(t)).$$

(B) Every new function symbol $\mathfrak{f}\colon M_{s'} \to M_s$ in $\mathfrak{F}$ is locally constant off of a finite union of strongly discrete sets; see Definition A.7.

Then $T(\mathfrak{F})$ is d-minimal at s_0.

Our criterion for d-minimality ensures that $\mathcal{L}(\mathfrak{F})$-terms are given locally by $\mathcal{L}$-definable functions off of a finite union of discrete sets. The following definition makes this more precise:

Definition 5.6. Suppose $\tau\colon M_0 \to M_s$ is an $\mathcal{L}(\mathfrak{F})$-term, with $s \in \mathcal{S}$. We define a **preparation** of τ to be a triple $(\boldsymbol{s}, B, f)$ consisting of a finite tuple of sorts $\boldsymbol{s}$, an $\mathcal{L}(\mathfrak{F})$-definable set $B \subseteq M_{\boldsymbol{s}} \times M_0$, and an $\mathcal{L}$-definable function $f\colon M_{\boldsymbol{s}} \times M_0 \to M_s$ such that:

(1) B_x is an open subset of M_0 for each $x \in M_{\boldsymbol{s}}$ and $B_x \cap B_{x'} = \varnothing$ for $x \neq x'$,
(2) $M_0 \setminus \bigcup_x B_x$ is a finite union of discrete sets,
(3) $\tau(t) = f_x(t)$ for each x and for all $t \in B_x$.

Moreover, if a preparation $(\boldsymbol{s}, B, f)$ of τ additionally satisfies:

(4) $f_x|_{B_x}: B_x \to M_s$ is continuous for each x,

then we say that $(\boldsymbol{s}, B, f)$ is a **continuous** preparation of τ.

Lemma 5.7. *Let $\tau: M_0 \to M_s$ be an $\mathcal{L}(\mathfrak{F})$-term. If τ has a preparation, then τ has a continuous preparation.*

Proof. Let $(\boldsymbol{s}, B, f)$ be a preparation of τ. For each x define $D_x \coloneqq \mathrm{cl}(\mathsf{Discont}(f_x|_{B_x}))$. Then there is $N \in \mathbb{N}$ such that $D_x \subseteq B_x$ is a union of N discrete sets for each x. It follows from Lemma A.6 that $D \coloneqq \bigcup_x D_x$ is a union of N discrete sets. Next, define $B^* \subseteq B$ by declaring for each $x \in M_{\boldsymbol{s}}$:

$$B^*_x \ \coloneqq\ B_x \setminus D_x \ =\ B_x \setminus D.$$

We claim that $(\boldsymbol{s}, B^*, f)$ is a continuous preparation. The definition of B^*_x ensures that (3) and (4) are satisfied. Since each D_x is closed, it follows that B^*_x is open, which is (1). Finally, for (2) note that:

$$M_0 \setminus \textstyle\bigcup_x B^*_x \ =\ (M_0 \setminus \bigcup_x B_x) \cup D,$$

which is a finite union of discrete sets. □

Lemma 5.8. *If τ is a variable, then $(\varnothing, M_0, \tau)$ is a continuous preparation of τ, where $\varnothing$ is the empty tuple of sorts.*

Lemma 5.9. *Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be a tuple from $\mathcal{S}$, let $s \in \mathcal{S}$, let $\sigma: M_{\boldsymbol{s}} \to M_s$ be an $\mathcal{L}$-term and let $\tau_i: M_0 \to M_{s_i}$ be an $\mathcal{L}(\mathfrak{F})$-term for each $i = 1, \ldots, n$. If each τ_i has a preparation, then the $\mathcal{L}(\mathfrak{F})$-term $\tau \coloneqq \sigma(\tau_1, \ldots, \tau_n)$ has a preparation.*

Proof. For each $i = 1, \ldots, n$, let $(\boldsymbol{s}_i, B_i, g_i)$ be a preparation of τ_i. Next, set $\bar{\boldsymbol{s}} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ and define $B \subseteq M_{\bar{\boldsymbol{s}}} \times M_0$ and $g: M_{\bar{\boldsymbol{s}}} \times M_0 \to M_s$ by declaring for all $x = (x_1, \ldots, x_n) \in M_{\boldsymbol{s}_1} \times \cdots \times M_{\boldsymbol{s}_n}$:

$$B_x \ \coloneqq\ B_{1,x_1} \cap \cdots \cap B_{n,x_n}, \qquad g(x,t) \ \coloneqq\ \sigma(g_1(x_1,t), \ldots, g_n(x_n,t)).$$

We claim that $(\bar{\boldsymbol{s}}, B, g)$ is a preparation of τ. Conditions (1) and (3) are clear. For (2), note that:

$$M_0 \setminus (\textstyle\bigcup_x B_x) \ =\ \bigcup_{1 \le i \le n}(M_0 \setminus \bigcup_{x_i} B_{i,x_i}),$$

which is a finite union of discrete sets. □

Lemma 5.10. *Suppose $\tau = \mathfrak{f}(\sigma)$ for some $\mathcal{L}(\mathfrak{F})$-term $\sigma: M_0 \to M_{s'}$ and some $\mathfrak{f}: M_{s'} \to M_s$ in $\mathfrak{F}$. Suppose also that:*

(1) σ has a preparation, and
(2) $\mathfrak{f}$ is locally constant off of a finite union of strongly discrete sets.

Then τ has a preparation.

Proof. Let $(\boldsymbol{s}, C, h)$ be a preparation of σ, which by Lemma 5.7 we may assume to be continuous. Define $B \subseteq M_{\boldsymbol{s}} \times M_s \times M_0$ by:

$$B_{x,y} \ \coloneqq\ \mathrm{int}\{t \in C_x : \mathfrak{f}(h_x(t)) = y\} \ \subseteq\ C_x$$

and define $f: M_{\boldsymbol{s}} \times M_s \times M_0 \to M_s$ by:

$$f(x,y,t) \ \coloneqq\ y.$$

We claim that $((\boldsymbol{s}, s), B, f)$ is a preparation of τ.

(1) By definition, each $B_{x,y}$ is open. Next suppose $t \in B_{x,y} \cap B_{x',y'}$. Then $t \in C_x \cap C_{x'}$, and thus $x = x'$ since $(\boldsymbol{s}, C, h)$ is a preparation. By definition of $B_{x,y}$, we have $y = \mathfrak{f}(h_x(t)) = y'$.

(2) As T is d-minimal at s_0, we find $N \in \mathbb{N}$ such that $\mathrm{br}(h_x^{-1}(z))$ is a union of N discrete sets for each $(x,z) \in M_{\boldsymbol{s}} \times M_{s'}$. Then Proposition A.19 gives $N' \in \mathbb{N}$ such that $C_x \setminus \bigcup_y B_{x,y}$ is a union

of N' discrete sets. Thus by Lemma A.6 we have that $\bigcup_x(C_x \setminus \bigcup_y B_{x,y})$ is a union of N' discrete sets as well. It follows that

$$M_0 \setminus \bigcup_{x,y} B_{x,y} \;=\; (M_0 \setminus \bigcup_x C_x) \;\cup\; \bigcup_x(C_x \setminus \bigcup_y B_{x,y})$$

is a finite union of discrete sets.

(3) Let x, y and $t \in B_{x,y}$ be arbitrary. Using that $(\boldsymbol{s}, C, h)$ is a preparation for σ and $t \in B_{x,y} \subseteq C_x$, we see that

$$\tau(t) \;=\; \mathfrak{f}(\sigma(t)) \;=\; \mathfrak{f}(h(x,t)) \;=\; y \;=\; f_{x,y}(t). \qquad \square$$

Corollary 5.11. *If each $\mathfrak{f} \in \mathfrak{F}$ is locally constant off of a finite union of strongly discrete sets, then every $\mathcal{L}(\mathfrak{F})$-term has a (continuous) preparation.*

We can now put together the material on preparations to prove Proposition 5.5.

Proof of Proposition 5.5. Let $D \subseteq M_0$ be $\mathcal{L}(\mathfrak{F})$-definable. By removing the interior of D (which is open and $\mathcal{L}(\mathfrak{F})$-definable), we may assume that D has empty interior. We will show that D is a finite union of discrete sets. By (A), there exist an n-ary $\mathcal{L}$-definable relation $R(x_1, \ldots, x_n)$ and $\mathcal{L}(\mathfrak{F})$-terms $\tau_1(t), \ldots, \tau_n(t)$ such that D is of the form:

$$D \;=\; \{t \in M_0 : R(\tau_1(t), \ldots, \tau_n(t))\}.$$

Next, for $i = 1, \ldots, n$, by (B) and Corollary 5.11 we take preparations $(\boldsymbol{s}_i, B_i, f_i)$ of each term τ_i. Set $\bar{\boldsymbol{s}} \coloneqq (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$ and define $B \subseteq M_{\bar{\boldsymbol{s}}} \times M_0$ by declaring for $x = (x_1, \ldots, x_n) \in M_{\boldsymbol{s}_1} \times \cdots \times M_{\boldsymbol{s}_n}$:

$$B_x \;\coloneqq\; B_{1,x_1} \cap \cdots \cap B_{n,x_n} \;\subseteq M_0.$$

Then each B_x is open and

$$M_0 \setminus \bigcup_x B_x \;=\; \bigcup_{1 \le i \le n}(M_0 \setminus \bigcup_{x_i \in M_{\boldsymbol{s}_i}} (B_i)_{x_i})$$

is a finite union of $\mathcal{L}(\mathfrak{F})$-definable discrete sets. We have

$$\begin{aligned} D \cap B_x \;&=\; \{t \in B_x : R_i\big(f_0(x_0,t), \ldots, f_{n-1}(x_{n-1},t)\big)\ \} \\ &=\; B_x \cap \underbrace{\{t \in M_0 : R_i\big(f_0(x_0,t), \ldots, f_{n-1}(x_{n-1},t)\big)\}}_{=:C_x}. \end{aligned}$$

Suppose $z \in \operatorname{int}(C_x)$, so there is an open U such that $z \in U \subseteq C_x$. Since $D \cap B_x$ does not have interior, it must be the case that $U \cap B_x = \varnothing$ (since B_x is open), thus $z \notin D \cap B_x$. In particular, $D \cap B_x = B_x \cap \operatorname{br}(C_x)$. By our assumption that T is d-minimal at s_0, $\operatorname{br}(C_x)$ is a union of N discrete sets, for some $N \in \mathbb{N}$ not depending on x, so $D \cap B_x$ is a union of N discrete sets.

By Lemma A.6, the set $D \cap \bigcup_{x \in M_{\bar{\boldsymbol{s}}}} B_x$ is a union of N discrete sets. Finally,

$$D \;=\; (D \cap \bigcup_{x \in M_{\bar{\boldsymbol{s}}}} B_x) \cup (D \cap (M_0 \setminus \bigcup_{x \in M_{\bar{\boldsymbol{s}}}} B_x))$$

is a finite union of discrete sets. $\square$

5.2. **Asymptotic couple.** Here, we describe how our d-minimality criterion can be used to quickly show that $T_{\log}$ is d-minimal. Let $(\Gamma, \psi) \models T_{\log}$. We take as a base theory T the theory of divisible ordered abelian groups with an infinite element, construed in the language

$$\mathcal{L} \;\coloneqq\; \{0, \infty, +, -, <, \delta_1, \delta_2, \delta_3, \ldots\}.$$

Then T is o-minimal, so assumptions (I) and (II) hold. Moreover $\mathcal{L}_{\log} = \mathcal{L} \cup \{\psi, s, p\}$, so $T_{\log}$ is an $\mathcal{L}_{\log}$-theory with quantifier elimination and a universal axiomatization. In particular, $T_{\log}$ satisfies condition (A) of Proposition 5.5. Lemma 2.8 gives that ψ, s, and p are all locally constant off of a discrete set (indeed, a strongly discrete set; see Lemma A.8), so condition (B) holds as well.

Remark 5.12. Using Corollary 5.11, one can easily prove the unary version of Corollary 3.17. That is, one can show that any definable function $f\colon \Gamma \to \Gamma_\infty$ is locally linear off of a finite union of discrete sets.

5.3. **Valued fields.** Let K be a field and let v be a henselian valuation on K of residue characteristic zero. We let $\boldsymbol{K}$ be the three-sorted structure $(K, \boldsymbol{k}, \Gamma; v, \mathrm{res})$, where $v\colon K^\times \to \Gamma$ and the residue map $\mathrm{res}\colon K \to \boldsymbol{k}$ are assumed to be surjective (and where the residue map sends everything with negative valuation to 0). We let $\mathcal{L}$ be the natural language of this three-sorted structure, but Morleyized on the sorts $\boldsymbol{k}$ and Γ (so $\mathcal{L}$ also includes an additional relation symbol for every $\varnothing$-definable subset of any cartesian power of $\boldsymbol{k}$ or Γ).

We view $\boldsymbol{K}$ as a topological $\mathcal{L}$-structure, where the sort K is equipped with the valuation topology and the sorts Γ and $\boldsymbol{k}$ are equipped with the discrete topology. We let T be the $\mathcal{L}$-theory of $\boldsymbol{K}$. Then T is a topological $\mathcal{L}$-theory with quantifier elimination. The models of T are precisely those henselian valued fields with residue field elementarily equivalent to $\boldsymbol{k}$ and value group elementarily equivalent to Γ.

Proposition 5.13. *The theory T satisfies assumptions (I) and (II), where the distinguished sort is the home sort.*

Proof. The valuation topology on K is T_1, and by [19, Proposition 5.1], every definable subset of K either has interior or is finite. For (II), Theorem 5.1.1 in [12] gives that every $\mathcal{L}$-definable function $K \to K$ is continuous on a dense open (hence cofinite) set. To verify that (II) holds for $\mathcal{L}$-definable functions $g\colon K \to \boldsymbol{k}$, we fix such a g and consider the $\mathcal{L}$-formula $\phi(x, y)$ defining g, so x is a variable of sort K and y is a variable of sort $\boldsymbol{k}$. Let $\tilde{\mathcal{L}}$ be the language containing sorts for $\boldsymbol{k}$ and Γ (as a field and an ordered abelian group, both Morleyized as above) together with a sort $\boldsymbol{k}/(\boldsymbol{k}^\times)^n$ and a corresponding quotient map $\pi_n\colon \boldsymbol{k} \to \boldsymbol{k}/(\boldsymbol{k}^\times)^n$ for each $n > 1$. By [2, Theorem 5.15], $\phi(x, y)$ is equivalent to a formula $\psi(\sigma_1(x), \ldots, \sigma_m(x), y)$ where ψ is an $\tilde{\mathcal{L}}$-formula and each σ_i is of the form $vP(x)$, $\mathrm{res}(P(x)/Q(x))$, or $\mathrm{res}^n(P(x))$, where $P, Q \in K[X]$ and where $\mathrm{res}^n\colon K \to \boldsymbol{k}/(\boldsymbol{k}^\times)^n$ is the map

$$\mathrm{res}^n(a) \;:=\; \begin{cases} 0 & \text{if } va \notin n\Gamma, \\ \pi^n \circ \mathrm{res}(a/b^n) & \text{if } va = nvb \text{ (this does not depend on choice of } b). \end{cases}$$

Since v, res, and the maps res^n are locally constant away from 0, each σ_i is locally constant off of a finite set. It follows that g is also locally constant (in particular, continuous) off of a finite set. The same argument works for $\mathcal{L}$-definable functions $h\colon K \to \Gamma$. □

We now extend $\mathcal{L}$ to $\mathcal{L}^* := \mathcal{L} \cup \{s, \ell\}$ and T to an $\mathcal{L}^*$-theory T^* with axioms stating that s is a section of the valuation and ℓ is a lift of the residue field. Note that not every model of T admits an expansion to a model of T^*, but if $\boldsymbol{K} \models T$ is $\aleph_1$-saturated, then $\boldsymbol{K}$ does admit such an expansion.

Corollary 5.14. *T^* is d-minimal (on the home sort).*

Proof. As a consequence of van den Dries's AKE theorem with lift and section [17, Section 5.3], the theory T^* eliminates quantifiers in the language $\mathcal{L}^*$; see also [32, Theorem 2.2]. Condition (A) of Proposition 5.5 follows. For condition (B), we note that both s and ℓ are locally constant everywhere, since Γ and $\boldsymbol{k}$ have the discrete topology. □

5.4. **The structure $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})^\#$.** Let $\tilde{\mathbb{R}}$ be a polynomially bounded expansion of the reals with field of exponents $\mathbb{Q}$, and let $\alpha > 1$.

It is easy to see how our criterion can be used to show that $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})$ is d-minimal (using the quantifier elimination result of Miller–van den Dries). Less easy is to see how d-minimality of $(\tilde{\mathbb{R}}, \alpha^{\mathbb{Z}})^\#$, the expansion of $\tilde{\mathbb{R}}$ by all subsets of all cartesian powers of $\alpha^{\mathbb{Z}}$, follows from our criterion, as the criterion only allows for new functions, not new predicates. In this subsection, we show how the multi-sorted setup can be used as a workaround.

Let $(\tilde{\mathbb{R}}, \mathbb{Z}^{\#})$ be a 2-sorted structure with underlying sets $\mathbb{R}$ and $\mathbb{Z}$. We construe this structure in the language $\mathcal{L}$ that contains:

(1) For each n, a function symbol for each function $\mathbb{R}^n \to \mathbb{R}$ that is definable without parameters in $\tilde{\mathbb{R}}$;

(2) the language of ordered abelian groups $\{0, +, <\}$ on $\mathbb{Z}$, along with a relation symbol for each subset of $\mathbb{Z}^n$.

Then $(\tilde{\mathbb{R}}, \mathbb{Z}^{\#})$ eliminates quantifiers in $\mathcal{L}$ and any substructure of $\tilde{\mathbb{R}}$ is an elementary substructure.

Now expand $(\tilde{\mathbb{R}}, \mathbb{Z}^{\#})$ by the map $\lambda\colon \mathbb{R}^{>} \to \mathbb{Z}$ sending $r \in \mathbb{R}^{>}$ to the least $z \in \mathbb{Z}$ with $\alpha^z \leq r < \alpha^{z+1}$. We extend λ to all of $\mathbb{R}$ by defining $\lambda(r) := 0$ for $r \leq 0$. Let $\mathcal{L}^* := \mathcal{L} \cup \{\lambda\}$. Note that λ is locally constant off of the set $\{0\} \cup \alpha^{\mathbb{Z}}$, which is a union of two strongly discrete sets. Thus, to prove d-minimality, it is enough to show that $(\tilde{\mathbb{R}}, \mathbb{Z}^{\#}; \lambda)$ has QE. This can be shown via a saturated embedding test, following the proof sketch given in [34, Section 8.6].

6. Final comments

6.1. The reader may have noticed that the pregeometries cl_ϕ are reminiscent of the so-called *small closure for lovely pairs* [8, Definition 4.5] and *dense pairs* [20, Definition 8.27]. We are unaware of any existing generalized pair framework that fits our situation exactly, although the following observations are in order:

- (Γ, Ψ) is a proper reduct of (Γ, ψ): using the embedding lemma [23, 4.6] one can change the ψ-value of a suitable archimedean class without changing the underlying Ψ-set.
- Ψ is a predicate that names an indiscernible sequence over Γ (in the language of ordered abelian groups with ∞); thus (Γ, Ψ) (and so definable sets of ∞-dimension zero, by Remark 3.23) may be further analyzed in the tradition of [7].
- (Γ, χ) is a proper reduct of (Γ, ψ), where $\chi(\alpha) := \int \psi(\alpha)$ for $\alpha \neq 0$ is the *contraction map* induced by the logarithm; this is proved in [24, Section 7.3].
- (Γ, χ, Ψ) is interdefinable with (Γ, ψ): given $\alpha \in \Gamma^{\neq}$, we may define $\psi(\alpha)$ to be the unique $\phi \in \Psi$ such that $\phi - s\phi = \chi(\alpha)$; here we use the restriction $s\colon \Psi \to \Psi$ of the successor function, which is definable in the reduct.
- We conjecture that χ (and thus ψ) is not definable in any monadic expansion of Γ.

6.2. The reader may have also noticed that for definable sets $X \subseteq \Gamma^n$ with $\dim_\infty(X) \leq 0$ (i.e., the "∞-very small sets") there are additional invariants available for a finer analysis. We record a few of them here:

- the least m such that X is a union of m discrete sets;
- the least m such that $X^{(m)} = \varnothing$;
- the least m such that X can be covered by finitely many affine maps of the form $\Psi^m \to \Gamma^n$;
- the least m such that there exists a definable surjection $\Psi^m \to X$;
- the dp-rank of X;
- when (Γ, ψ) is the standard model, we may associate to X the (growth rate of the) function $\mathbb{N} \to \mathbb{N} : k \mapsto |\pi_{s^k 0}(X)|$, which we conjecture is always eventually equal to a polynomial with rational coefficients. For example, with the $X \subseteq \Gamma$ described in the beginning of Subsection 3.3, we have $|\pi_{s^k 0}(X)| = \binom{k-1}{2} + \binom{k-1}{1} + \binom{k-1}{0} = \frac{1}{2}k^2 - \frac{1}{2}k + 1$.

We make no general claims about these invariants, although some basic relations between these quantities can be read off from our results in Section 4.

6.3. Finally, we address some possible extensions and limitations of this work, in particular as it compares to analogous results for (the asymptotic couple of) the differential field $\mathbb{T}$ of *logarithmic-exponential transseries* by Aschenbrenner, van den Dries, and van der Hoeven. Note that their main [5, Theorem 0.1] can be viewed as a "Small Sets Theorem" for that setting.

It is clear that $(\Gamma_{\log}, \psi)$ is part of $\mathbb{T}^{\mathrm{eq}}_{\log}$. However, the induced structure on $\Gamma_{\log}$ is more than that of the 1-sorted asymptotic couple $(\Gamma_{\log}, \psi)$. In fact, the following binary map is also definable in $\mathbb{T}^{\mathrm{eq}}_{\log}$:

$$\mathrm{sc}\colon \mathbb{R} \times \Gamma_{\log} \to \Gamma_{\log}, \quad \mathrm{sc}(r, \gamma) \;=\; r\gamma$$

and consequently we also should consider the 2-sorted strict expansion $\boldsymbol{\Gamma}_{\mathbb{T}_{\log}} = ((\Gamma_{\log}, \psi), \mathbb{R}; \mathrm{sc})$; this is for the same reason as given in [5], namely that the constant power "map" on $\mathbb{T}_{\log}$ induces a scalar multiplication by the constant/residue field $\mathbb{R}$ on the value group $\Gamma_{\log}$. Of course, the 2-sorted setting introduces new types of discrete sets that are not captured by the Small Sets Theorem, such as $\mathbb{R}s0 := \mathrm{sc}(\mathbb{R}, s0)$, but we expect them to be "orthogonal" to the discrete set Ψ in a relevant sense.

As in [5], we also know that for nonzero differential polynomials $G(Y) \in \mathbb{T}_{\log}\{Y\}$ the subset $\{vy : y \in \mathbb{T}^{\times}_{\log} : G(y) = 0\}$ of $\Gamma_{\log}$ is discrete since [3, Corollaries 14.3.10, 14.3.11] also apply to $\mathbb{T}_{\log}$. For example, the differential polynomial $G(Y) = x(YY'' - (Y')^2) + YY'$ yields the discrete set $\mathbb{R}s0$ in $\Gamma_{\log}$; this is because the nonzero zeros of $G(Y)$ satisfy $(xY^{\dagger})' = 0$, i.e., $Y^{\dagger} \in \mathbb{R}x^{-1}$. Conversely, certain discrete sets like Ψ cannot occur of the form "$v(Z(G)^{\neq})$" by [3, Corollary 13.4.5].

We do not know whether $\boldsymbol{\Gamma}_{\mathbb{T}_{\log}}$ is purely stably embedded in $\mathbb{T}^{\mathrm{eq}}_{\log}$: a positive answer would require an elimination theory for $\mathbb{T}_{\log}$, and a negative answer would require a counterexample. This was answered in the negative for $\mathbb{T}$ in [5, p. 532]. However, their counterexample produces an injection $c \mapsto [v(e^{e^{cx}})] : \mathbb{R}^{>} \to [\Gamma^{\neq}_{\mathbb{T}}]$ into the archimedean classes of $\Gamma_{\mathbb{T}}$; such a phenomenon is impossible in our setting since $[\Gamma^{\neq}_{\log}] \cong \Psi$ is countable.

Appendix A. Topology

Our main theorems concern the agreement (or disagreement) of various ideals of definable sets. In this appendix, we recall some basic facts we need about those ideals that arise from topology. In particular, we observe that under mild topological assumptions (e.g., if X is T_1 with no isolated points), we get a linear inclusion of ideals:

$$\text{finite} \;\subseteq\; \text{d-finite} \;\subseteq\; \text{finite unions of discrete sets} \;\subseteq\; \text{nowhere dense}$$

(See Lemmas A.13, and A.5.) Furthermore, in Subsection A.6 we establish Proposition A.19, which is needed for the d-minimality criterion 5.5.

Throughout, X and Y are topological spaces and we let A, B range over subsets of X.

A.1. **Preliminaries.** We denote the **interior** and **closure** of A in X by $\mathrm{int}(X)$ and $\mathrm{cl}(X)$. We may use subscripts if we want to emphasize the ambient space. For example:

Lemma A.1. *If $f\colon X \to Y$ is continuous, then for any $A \subseteq X$ we have $\mathrm{cl}_X(A) \subseteq f^{-1}(\mathrm{cl}_Y(f(A)))$, equivalently, $f(\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A))$.*

We occasionally make the following assumptions about our topological space:

Definition A.2. We say that X is T_1 if every singleton is closed, and we say that X has **no isolated points** if every singleton is not open.

A.2. **Sets with empty interior.** We say a set A has **empty interior in** X if $\mathrm{int}_X(A) = \varnothing$. When the ambient space X is understood from context, we just say A has **empty interior**. Conversely, we say A **has interior** if A does not have empty interior.

The sets with empty interior in general do not form an ideal; moreover, the ideal that they generate is often the improper ideal (e.g., consider $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ with the usual topology.)

A.3. **The ideal of nowhere dense sets.** We say A is **nowhere dense in** X if $\mathrm{int}_X(\mathrm{cl}_X(A)) = \varnothing$. When the ambient space X is understood from context, we just say A is **nowhere dense**.

The nowhere dense subsets of X form an ideal of subsets of X, which is closed under taking closures:

Lemma A.3 (Nowhere dense sets form an ideal)**.**
(1) A finite union of nowhere dense sets is nowhere dense.
(2) If $A \subseteq B$ and B is nowhere dense, then A is nowhere dense.
(3) A is nowhere dense iff $\mathrm{cl}(A)$ is nowhere dense.

The following is obvious:

Lemma A.4. *If A is nowhere dense, then A has empty interior.*

A.4. **The ideal generated by discrete sets.** Recall that a set A is **discrete** (in X) if A is a discrete space when equipped with the subspace topology.

In general, the discrete subsets of X do not form an ideal (consider $\{0\} \cup \{1/n : n \geq 1\}$ in $\mathbb{R}$ with the usual topology), although really we are interested in the ideal generated by the discrete sets, i.e., the sets that are finite unions of discrete sets.

We first observe that under mild topological assumptions, the ideal generated by discrete sets is sandwiched between the ideal of finite sets and the ideal of nowhere dense sets:

Lemma A.5. *Suppose X is T_1 and has no isolated points. Then for every $D \subseteq X$ we have:*

$$D \text{ is finite} \quad \Rightarrow \quad D \text{ is discrete} \quad \Rightarrow \quad D \text{ is nowhere dense.}$$

Proof. The first implication is immediate from T_1. Now assume D is discrete and assume towards a contradiction that $\mathrm{int}(\mathrm{cl}(D)) \neq \varnothing$. Then there exists a nonempty open set $U \subseteq \mathrm{cl}(D)$. Note that $U \cap D \neq \varnothing$ and fix $d \in U \cap D$. Since D is discrete, we can choose an open set $V \subseteq U$ such that $V \cap D = \{d\}$. Next, next note that $W := V \setminus \{d\}$ is open (by T_1) and nonempty (since X has no isolated points). Moreover, we have $W \subseteq \mathrm{cl}(D)$, however $W \cap D = \varnothing$, a contradiction. $\square$

In general both directions $\Leftarrow$ can fail. Indeed, the second is the distinction between the Very Small Sets and the Kinda Small Sets when $n > 1$.

Sets in the ideal generated by discrete sets can be glued together in the following obvious way:

Lemma A.6. *Suppose:*
(1) $(B_i)_{i \in I}$ is a disjoint family of open sets,
(2) $(D_i)_{i \in I}$ is a family with $D_i \subseteq B_i$ for each $i \in I$,
(3) there is $N \in \mathbb{N}$ such that each D_i is a union of N discrete sets.
Then $\bigsqcup_{i \in I} D_i$ is a union of N discrete sets.

When applying this lemma, the following concept is useful:

Definition A.7. We say a set $D \subseteq X$ is **strongly discrete** if there exists a family $(V_d)_{d \in D}$ of pairwise disjoint open sets such that $V_d \cap D = \{d\}$ for each $d \in D$.

In many topological spaces of interest, we get *strongly discrete* for free:

Lemma A.8. *Suppose X is either (i) a metric space equipped with the metric topology, or (ii) a valued abelian group equipped with the valuation topology. Then for every $D \subseteq X$ we have: D is discrete iff D is strongly discrete.*

Note the above lemma can fail, even for hausdorff spaces. Indeed, in the *Niemytzki's tangent disk topology* [38, II.82] on $X = \mathbb{R} \times [0, +\infty)$ the axis $\mathbb{R} \times \{0\}$ is discrete but not strongly discrete.

A.5. **The ideal of d-finite sets.** Given $x \in X$ we say that x is a **limit point** of A (in X) if for every neighbourhood U of x:

$$(U \setminus \{x\}) \cap A \;\neq\; \varnothing.$$

Define the **derived set** of A (in X) to be:

$$A' \;\coloneqq\; \{x \in X : x \text{ is a limit point of } A\}.$$

Note that the derived set A' of A depends on the ambient space X. For $n \geq 0$ we define $A^{(n)}$ recursively by setting $A^{(0)} \coloneqq A$ and $A^{(n+1)} \coloneqq (A^{(n)})'$.

Here are some basic facts about derived sets:

Lemma A.9. *Given A, B and a finite family $(A_i)_{i \in I}$ of subsets of X we have for every $n \geq 0$:*

(1) $(\bigcup_{i \in I} A_i)^{(n)} = \bigcup_{i \in I} A_i^{(n)}$.
(2) $A^{(n)} \subseteq A \cup A' = \operatorname{cl}(A)$*; in particular,* $\operatorname{cl}(A)^{(n)} = A^{(n)} \cup A^{(n+1)} = \operatorname{cl}(A^{(n)})$.
(3) If $A \subseteq B$, then $A^{(n)} \subseteq B^{(n)}$.
(4) A is discrete and closed if and only if $A' = \varnothing$.

In particular, if A is closed, then $(A^{(n)})_{n \geq 0}$ is a decreasing sequence of closed sets. Moreover, if X is T_1, then $A' = \operatorname{cl}(A)'$, hence $(A^{(n)})_{n \geq 1}$ is a decreasing sequence of closed sets. However, this behavior need not be typical in general:

Example A.10. Suppose $X = \{a, b\}$ with the indiscrete topology. Then for $A = \{a\}$ we have:

$$A^{(n)} \;\coloneqq\; \begin{cases} \{a\} & \text{if } n \text{ even}, \\ \{b\} & \text{if } n \text{ odd}. \end{cases}$$

Thus in general $A^{(n)}$ need not be closed, the sequence $(A^{(n)})_{n \geq 0}$ need not be eventually decreasing, and $\operatorname{cl}(A)^{(n)}$ need not equal $A^{(n)}$.

We introduce the following terminology:

Definition A.11. We say that A is **d-finite in** X if $A^{(n)} = \varnothing$ for some $n \geq 0$. When the ambient space X is understood from context, we just say A is **d-finite**.

The d-finite subsets of X form an ideal of subsets of X, which is closed under taking closures:

Lemma A.12 (d-finite sets form an ideal)**.** *For every A, B we have:*

(1) A finite union of d-finite sets is d-finite.
(2) If $A \subseteq B$ and B is d-finite, then A is d-finite.
(3) A is d-finite iff $\operatorname{cl}(A)$ is d-finite.

Proof. (1) follows from A.9(1), (2) follows from A.9(3), and (3) follows from A.9(2). □

The ideal of d-finite sets is always contained in the ideal generated by discrete sets:

Lemma A.13. *For every A, the difference $A \setminus A'$ is discrete; hence, if A is d-finite, then A is a finite union of discrete sets.*

In general the converse can fail in a T_1 space:

Example A.14. Consider the ordinal $X \coloneqq \omega_1$ equipped with the order topology, and the subset $D \coloneqq \{\alpha+1 : \alpha \in \omega_1\} \subseteq X$ of successor ordinals. Then D is discrete, although D is not d-finite.

Here is a T_1 example without any isolated points:

Example A.15. Consider the product $X = \mathbb{R} \times \mathbb{R}$. Equip each point $(r, 0)$ with the usual neighbourhood basis inherited from the standard topology on $\mathbb{R}^2$. Equip each point (r, x) with $x \neq 0$ with the neighbourhood basis of vertical intervals of the form $\{(r,t) : x - \varepsilon < t < x + \varepsilon\}$, where $0 < \varepsilon < |x|$. Consider the set $D \coloneqq \mathbb{R} \times \{1/n : n > 0\}$. Then D is discrete, although $D^{(n)} = \mathbb{R} \times \{0\}$ for every $n \geq 1$. Thus D is not d-finite.

Remark A.16 (Relation to Cantor–Bendixson derivative)**.** The *Cantor–Bendixson (CB) derivative* is initially defined for topological spaces X as $d_{\mathrm{CB}}(X) \coloneqq X'$ as in [31, (6.10)], although one often extends this definition to arbitrary subsets $A \subseteq X$ via $d_{\mathrm{CB}}(A) \coloneqq A \setminus \operatorname{isol}(A)$ (as done in [21, 3.28] for hausdorff spaces), where $\operatorname{isol}(A)$ is the set of isolated points of A in the subspace topology induced by X. In a T_1-space, we have $d^n_{\mathrm{CB}}(A) = \varnothing$ if and only if A is a union of at most n discrete sets, which shows that this property is definable in families; see [21, 3.29].

Following the terminology from [31], we say that A has *finite CB-rank* if the sequence $(d^n_{\mathrm{CB}}(A))_{n\geq 0}$ is eventually constant, in which case we call the eventual value the *perfect kernel* of A. It then follows that the property "A is d-finite" is equivalent to "$\operatorname{cl}(A)$ has finite CB-rank with empty perfect kernel". Note that if A is d-finite, then A has finite CB-rank (with empty perfect kernel), although the converse need not hold as shown in Examples A.14 and A.15. We could not find an existing name for the property *d-finite* in the literature.

The following implies that the product of d-finite sets is again d-finite in the product topology:

Lemma A.17. *Suppose $A \subseteq X$ and $C \subseteq Y$ are closed. For $k \in \mathbb{N}$, we have*

$$(A \times C)^{(k)} \;=\; \textstyle\bigcup_{m+n=k} (A^{(m)} \times C^{(n)}),$$

where $A \times C$ is considered as a subset of the product space $X \times Y$. In particular, if $A^{(m)} = C^{(n)} = \varnothing$ for some $m, n \in \mathbb{N}$ with $m + n > 0$, then $(A \times C)^{(m+n-1)} = \varnothing$.

Proof. The $k = 1$ case $(A \times C)' = (A' \times C) \cup (A \times C')$ is routine. The general case follows by induction on k. □

A.6. **Border and locally constant functions.** Given a set $A \subseteq X$, we let $\operatorname{br} A \coloneqq A \setminus \operatorname{int} A$ denote the **border** of A. The *border* of A should not be confused with the *boundary* of A ($= \operatorname{cl}(A) \setminus \operatorname{int}(A)$) or the *frontier* of A ($= \operatorname{cl}(A) \setminus A$); we do not use boundary or frontier in this paper.

We mainly use border to detect where a function is (not) locally constant: Suppose $\mathfrak{f} \colon Y \to Z$ is an arbitrary function. Then we have:

$$\{y \in Y : \mathfrak{f} \text{ is locally constant at } y\} \;=\; \textstyle\bigsqcup_{z \in Z} \operatorname{int} \mathfrak{f}^{-1}(z),$$

which is always an open set; we also have:

$$\{y \in Y : \mathfrak{f} \text{ is not locally constant at } y\} \;=\; \textstyle\bigsqcup_{z \in Z} \operatorname{br} \mathfrak{f}^{-1}(z),$$

which is always a closed set, being the complement of the first set.

We conclude with Proposition A.19 below, which is a technical membership criterion for this ideal that we use in the proof of Proposition 5.5. First a lemma:

Lemma A.18. *Suppose $h \colon X \to Y$ is continuous and $D \subseteq Y$ is arbitrary. Then:*

$$\operatorname{int} h^{-1}D \;\supseteq\; \textstyle\bigsqcup_{d \in D} \operatorname{int} h^{-1}(d) \quad \textit{and thus} \quad \operatorname{br} h^{-1}D \;\subseteq\; \textstyle\bigsqcup_{d \in D} \operatorname{br} h^{-1}(d).$$

Proof. For $d \in D$ we have $h^{-1}(d) \subseteq h^{-1}(D)$, thus $\operatorname{int} h^{-1}(d) \subseteq \operatorname{int} h^{-1}(D)$, which yields the first inclusion. The second inclusion follows from taking complements inside $h^{-1}(D)$. □

Proposition A.19. *Let $h\colon X \to Y$ be a continuous function, and $\mathfrak{f}\colon Y \to Z$ be a function. Suppose:*

(1) $\mathfrak{f}\colon Y \to Z$ is locally constant outside a union of M strongly discrete sets, and

(2) there is $N \in \mathbb{N}$ such that $\operatorname{br} h^{-1}(y)$ is a union of N discrete sets for every $y \in Y$.

Then

$$X \setminus \bigsqcup_z \operatorname{int} X_z \;=\; \bigsqcup_z \operatorname{br} X_z$$

is the union of MN discrete sets, where $X_z := h^{-1}\mathfrak{f}^{-1}(z)$, i.e., the composition $\mathfrak{f} \circ h\colon X \to Z$ is locally constant outside a union of MN discrete sets.

Proof. Set $D := \bigsqcup_{z\in Z} \operatorname{br}(\mathfrak{f}^{-1}(z))$. By assumption, we may take a partition $(D^i)_{1\le i\le M}$ of D into M strongly discrete sets D^i.

For $z \in Z$ define $U_z := \operatorname{int}\mathfrak{f}^{-1}(z)$ and $D_z := \operatorname{br}\mathfrak{f}^{-1}(z)$. Note that our assumption on $\mathfrak{f}$ says that $D = \bigsqcup_z D_z$, and hence each D_z is a union of the M discrete sets $D^i_z := D_z \cap D^i$ where $i = 1, \dots, M$; hence $D^i = \bigsqcup_{z\in Z} D^i_z$. Thus we have a partition:

$$\mathfrak{f}^{-1}(z) \;=\; U_z \sqcup D_z \;=\; U_z \sqcup \bigsqcup_{i=1}^M D^i_z.$$

Pulling this back along the continuous function $h\colon X \to Y$ yields:

$$X_z \;=\; h^{-1}\mathfrak{f}^{-1}(z) \;=\; h^{-1}(U_z) \sqcup \bigsqcup_{i=1}^M h^{-1}(D^i_z) \;=\; \underbrace{h^{-1}(U_z) \sqcup \bigsqcup_{i=1}^M \operatorname{int} h^{-1}(D^i_z)}_{\text{open}} \sqcup \bigsqcup_{i=1}^M \operatorname{br} h^{-1}(D^i_z).$$

Hence we have $\operatorname{br} X_z \subseteq \bigsqcup_{i=1}^M \operatorname{br} h^{-1}(D^i_z) \subseteq D_z$ for each $z \in Z$. Thus by Lemma A.18 we have:

$$\bigsqcup_z \operatorname{br} X_z \;\subseteq\; \bigsqcup_{z\in Z} \bigsqcup_{i=1}^M \operatorname{br} h^{-1}(D^i_z) \;=\; \bigsqcup_{z\in Z} \bigsqcup_{i=1}^M \bigsqcup_{d\in D^i_z} \operatorname{br} h^{-1}(d) \;=\; \bigsqcup_{i=1}^M \bigsqcup_{d\in D^i} \operatorname{br} h^{-1}(d).$$

It suffices to show for each i that $\bigsqcup_{d\in D^i} \operatorname{br} h^{-1}(d)$ is a finite union of N discrete sets. Take for each $d \in D^i$ an open $V_d \subseteq Y$ such that $V_d \cap D^i = \{d\}$; moreover, since D^i is strongly discrete, we may assume that the family $(V_d)_{d\in D^i}$ is pairwise disjoint. Then $(h^{-1}V_d)_{d\in D^i}$ is a disjoint family of open sets with $\operatorname{br} h^{-1}(d) \subseteq h^{-1}V_d$ for each $d \in D^i$, so $\bigsqcup_{d\in D^i} \operatorname{br} h^{-1}(d)$ is a union of N discrete sets by Lemma A.6. □

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