GENERIC DERIVATIONS, DIFFERENTIAL LARGENESS, AND NTP_2

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ABSTRACT. We compare Fornasiero and Terzo's framework of generic derivations on algebraically bounded structures with León Sánchez and Tressl's differentially large fields. We show in the case of a single derivation that genericity and differential largeness coincide for éz-fields, as introduced by Walsberg and Ye. We also show that an NTP_2 algebraically bounded structure remains NTP_2 after expanding by a generic derivation.

1. INTRODUCTION

In this note, $\mathcal{L}$ is a language extending $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$, and K is an $\mathcal{L}$-structure expanding a field of characteristic zero. We let n, m, k, r range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let dim denote algebraic dimension on subsets of cartesian powers of K, so $\dim X$ is the dimension of the Zariski closure of $X \subseteq K^n$.

The last few years have seen new approaches to the model theory of "tame" differential fields, often under the guiding principle: "The model theory of a differential field is completely determined by the model theory of the underlying field, so long as the derivation is generic." *Genericity* roughly means that anything that can happen, subject to the constraints imposed by differential algebra and the theory of the underlying field, does happen.

One recent approach is due to Fornasiero and Terzo [8], who axiomatized what it means for a derivation on an algebraically bounded expansion of a field to be *generic*. The structure K is **algebraically bounded** if for all elementary extensions $K^* \succcurlyeq_{\mathcal{L}} K$, all $B \subseteq K^*$, and all $a \in K^*$, we have

$$a \in \operatorname{acl}_{\mathcal{L}}(K \cup B) \iff \operatorname{trdeg}(a|K(B)) = 0.$$

Many tame classes of fields are known to be algebraically bounded, including real closed fields, algebraically closed fields, and henselian fields of characteristic zero. If K is algebraically bounded, then dim is a definable dimension on K, meaning that for any definable family $(X_a)_{a\in A}$ and any d, the set of $a \in A$ for which $\dim(X_a) = d$ is definable. Algebraic boundedness was introduced by van den Dries [23], and the equivalence of van den Dries' definition with the one given here is [13, Lemma 2.12].

Let δ be a derivation on K. Fornasiero and Terzo define genericity as follows:

Definition 1.1 (Genericity)**.** The derivation δ is **generic** if for all $\mathcal{L}(K)$-definable $X \subseteq K^{1+r}$, if the projection of X onto the first r coordinates has dimension r, then there is $a \in K$ with $(a, \delta a, \ldots, \delta^r a) \in X$.

When K is algebraically bounded, genericity is first-order axiomatizable. Fornasiero and Terzo go on to show that if K is algebraically bounded and δ is generic, then several model-theoretic properties of K transfer to (K, δ), including quantifier elimination and model completeness.

Another recent research program, initiated by León Sánchez and Tressl, is the study of *differentially large fields* [17]. These fields serve as a differential analog of the *large* fields of Pop; see [21]. Unlike the setting of algebraically bounded structures with generic derivations, differential largeness is defined only for *pure* differential fields (that is, when $\mathcal{L} = \mathcal{L}_{\text{ring}}$).

Definition 1.2 (Differential largeness)**.** The underlying differential field (K, δ) is **differentially large** if

(1) K is large as a field, and
(2) for every differential field extension $(L, \delta) \supseteq (K, \delta)$, if K is existentially closed in L as a field, then (K, δ) is existentially closed in (L, δ) as a differential field.

What is the relationship between these two notions? We note that there are large, non-algebraically bounded fields [6, Example 10], as well as algebraically bounded, non-large fields [14, Example 4.30]. Using the characterization of differential largeness provided in [18], it is not difficult to show that if the underlying

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field K is large and δ is generic, then the underlying differential field (K,δ) is differentially large; see Corollary 2.4. As for the converse, an obvious obstruction occurs in the case that there are $\mathcal{L}$-definable subsets of K that are not definable in the $\mathcal{L}_{\mathrm{ring}}$-language, as the axiom of differential largeness can't assert anything about these sets. Here is a concrete example:

Example 1.3. Let (K,δ) be a differentially closed field and let $C := \ker(\delta)$ be the constant field of K. Let $t \in K$ be transcendental over C and consider the subfield $C(t)$ of K. Let $\mathcal{O}_t \subseteq C(t)$ be the t-adic valuation ring on $C(t)$, and let $\mathcal{O} \subseteq K$ be a valuation ring lying over $\mathcal{O}_t$ (we can even find $\mathcal{O}$ with residue field isomorphic to C; see [11, Chapter V, Theorem 9]). Then $(K,\mathcal{O})$ is an algebraically closed nontrivially valued field. Let $\mathcal{L} = \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ be the language of valued fields, so $(K,\mathcal{O})$ is algebraically bounded as an $\mathcal{L}$-structure. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. Then $\dim(\mathfrak{m}) = 1$, but there is no $a \in \mathfrak{m}$ with $\delta a = 0$, since $C \subseteq \mathcal{O}^\times = \mathcal{O} \setminus \mathfrak{m}$. Thus, δ is not generic.

In light of this example, a converse only becomes plausible if we assume that K is a large field with no additional structure.

Question. *Suppose that $\mathcal{L} = \mathcal{L}_{\mathrm{ring}}$. If (K,δ) is differentially large, then is δ generic?*

It seems difficult to answer this question without some understanding of the definable sets in K. In Proposition 2.3 below, we give a "topological" axiomatization of differential largeness in terms of the étale open topology of Johnson, Tran, Walsberg, and Ye [12]. In [24], Walsberg and Ye isolated the class of **éz-fields**—large, algebraically bounded fields that are "topologically tame" with respect to this topology. For these fields, we have a good enough understanding of definable sets to answer the question positively.

Theorem A (Corollaries 2.4 and 2.5)**.** *Suppose that $\mathcal{L} = \mathcal{L}_{\mathrm{ring}}$ and that K is an éz-field. Then δ is generic if and only if (K,δ) is differentially large.*

Theorem A was known in the case that K is large and model complete (possibly after adding constants), as an easy consequence of [17, Proposition 4.8]. This in turn uses that differentially large fields can be axiomatized via Tressl's "uniform companion" [22]. These fields, along with many others (including henselian fields of characteristic zero and perfect Frobenius fields), are all éz [24, Theorem C].

In the second part of this note, we focus on model-theoretic transfer theorems for algebraically bounded structures with generic derivations. Suppose that K is algebraically bounded and let T be the $\mathcal{L}$-theory of K. Let $\mathbb{P} := \mathrm{dcl}_{\mathcal{L}}(\emptyset)$ and fix a derivation $\delta_{\mathbb{P}}$ on $\mathbb{P}$ (one could, for instance, take $\delta_{\mathbb{P}}$ to be the zero derivation; when $\mathbb{P}$ is algebraic over $\mathbb{Q}$, this is the only possibility). Let $\mathcal{L}^\delta = \mathcal{L} \cup \{\delta\}$, let $T^\delta = T +$ "δ is a derivation extending $\delta_{\mathbb{P}}$", and let $T^\delta_g = T^\delta +$ "δ is generic". We include a list of previously established transfer theorems below.

Fact 1.4.

1. *T^δ_g is consistent and complete.*
2. *For every $\mathcal{L}^\delta$-formula $\varphi(x)$ (x is a tuple of variables), there is an $\mathcal{L}$-formula $\psi(x_0,\ldots,x_r)$ such that*
$$T^\delta_g \models \forall x(\varphi(x) \leftrightarrow \psi(x,\delta x,\ldots,\delta^r x)).$$
3. *If T is model complete, then T^δ_g is the model companion of T^δ.*
4. *If T eliminates quantifiers, then so does T^δ_g.*
5. *If T is stable, then so is T^δ_g.*
6. *If T has NIP, then so does T^δ_g.*
7. *If T is distal, then so is T^δ_g.*
8. *If T is simple, then so is T^δ_g.*
9. *If T^δ_g eliminates imaginaries, then it is rosy.*
10. *If T has NSOP$_1$, then so does T^δ_g.*

The first nine parts of Fact 1.4 were shown by Fornasiero and Terzo; see [8] for (1)–(6) and [9] for (7)–(9). Part (10), as well as independent proofs of (5), (8), and (9), can be established using León Sánchez and Mohamed's framework of *derivation-like theories* [16]. Transfers of neostability properties defined in terms of sequences, like (5)–(7), can be established quite easily using (2); see [3, Proposition 7.1] and [8, Theorem 6.2] for general criteria, as well as the proofs in [5, 7]. For properties that can be defined in terms of independence relations like (8)–(10), one shows that an independence relation on T can be used to define one on T_g^δ satisfying the same properties; see [8, 16]. A third class of model-theoretic properties consists of *tree properties*, which are defined by consistency-inconsistency patterns. As far as we are aware, the only transfer result for these types of properties without using independence relations is due to Point [20], who shows that NTP_2 transfers for certain classes of topological fields with generic derivations [10]. We generalize this to all algebraically bounded fields. Our method relies on certain facts about NTP_2-theories, such as Chernikov's one-variable theorem, but is general enough to be applied to other tree properties, such as the *antichain tree property* [2].

Theorem B (Theorems 3.1 and 3.2)**.** *If T has NTP_2, then so does T_g^δ. If T has NATP (that is, if T doesn't have the antichain tree property), then so does T_g^δ.*

Differentially large fields and generic derivations are both defined in the case of finitely many commuting derivations, and the results in Fact 1.4 hold in this more general setting. It seems quite plausible that our Theorems A and B also hold in this setting as well, but we do not investigate this here.

2. Topological axioms and éz-fields

In giving topological axioms for differential largeness, we need the following alternative axiomatization in terms of differential polynomials:

Fact 2.1 ([18, Theorem 2.8])**.** *A differential field (K, δ) is differentially large if and only if*

(1) K is large as a field and
(2) for all $r > 0$, all $P \in K[X_0, \dots, X_r]$, and all nonzero $Q \in K[X_0, \dots, X_{r-1}]$, if there is $x \in K^{1+r}$ with $P(x) = 0$ and $\frac{\partial P}{\partial X_r}(x) \neq 0$, then there is $a \in K$ with

$$P(a, \delta a, \dots, \delta^r a) = 0 \neq Q(a, \delta a, \dots, \delta^{r-1} a).$$

Let V be a K-variety and let $V(K)$ denote the set of K-points of V. The **étale open topology** or **$\mathcal{E}_K$-topology** on $V(K)$ is the topology with basis given by sets of the form $f(W(K))$ for étale morphisms $f\colon W \to V$. Equipping the K-points of each K-variety with the $\mathcal{E}_K$-topology, we obtain a *system of topologies* [12, Theorem A], meaning that morphisms $f\colon V \to W$ between K-varieties induce continuous maps $V(K) \to W(K)$ with respect to the $\mathcal{E}_K$-topologies, and that these induced maps are open (resp. closed) embeddings whenever f is an open (resp. closed) immersion. The field K is **large** if and only if the topology on $V(K)$ is non-discrete whenever $V(K)$ is infinite [12, Theorem C]. For our purposes, we can take this as a definition of largeness. An $\mathcal{L}_{\mathrm{ring}}(K)$-definable set $X \subseteq V(K)$ is **éz** if X is a finite union of étale open subsets of Zariski closed subsets of $V(K)$. If K is not large, then any $\mathcal{L}_{\mathrm{ring}}(K)$-definable set is éz. For large fields, the class of éz sets is quite well-behaved:

Fact 2.2 ([24, Theorems A and B(2)])**.** *Suppose K is large (and perfect, but we always assume characteristic zero in this note).*

(1) The class of éz sets is closed under morphisms of K-varieties; in particular, all existentially $\mathcal{L}_{\mathrm{ring}}(K)$-definable sets are éz.
(2) For V a smooth irreducible K-variety and $X \subseteq V(K)$ a nonempty éz set, we have $\dim X = \dim V$ if and only if X has nonempty $\mathcal{E}_K$-interior in $V(K)$.

Now let δ be a derivation on K. For a variety V, we let τV denote the **prolongation** of V, and we let π_V denote the projection map $\tau V \to V$; see [19]. The prolongation is an analog of the tangent bundle TV that takes the derivatives of defining parameters into account: if $x \in V(K)$, then $\delta x \in (\tau_x V)(K)$, and when V is defined over the constant field $\ker(\delta)$, the prolongation and tangent bundle coincide. For $a = (a_1, \dots, a_n) \in K^n$, we let $\delta a \coloneqq (\delta a_1, \dots, \delta a_n)$, and for $r \in \mathbb{N}$, we let $\nabla^r(a) \coloneqq (a, \delta a, \dots, \delta^r a) \in K^{(1+r)n}$.

Proposition 2.3. *Suppose that K is a large field and let δ be a derivation on K. The following are equivalent:*

(1) For every smooth irreducible K-variety V and every éz set $X \subseteq (\tau V)(K)$, if $\pi_V(X) \subseteq V(K)$ has $\mathcal{E}_K$-interior, then there is $a \in V(K)$ with $(a, \delta a) \in X$.
(2) For every éz set $X \subseteq K^{2r}$, if $\pi(X) \subseteq K^r$ has $\mathcal{E}_K$-interior, then there is $a \in K^r$ with $(a, \delta a) \in X$.

(3) *For every éz set $X \subseteq K^{r+1}$, if $\pi(X) \subseteq K^r$ has $\mathcal{E}_K$-interior, then there is $y \in K$ with $\nabla^r(y) \in X$.*
(4) *(K, δ) is differentially large.*

Proof. For (1)⇒(2), just take $V = \mathbb{A}^r$. Suppose (2) holds and let $X \subseteq K^{1+r}$ be as in (3). Consider the morphism $f: \mathbb{A}^{1+r} \to \mathbb{A}^{2r}$ given by

$$(x_0, \dots, x_r) \mapsto (x_0, \dots, x_{r-1}, x_1, \dots, x_r).$$

Then $f(X) \subseteq K^{2r}$ is éz by Fact 2.2(1) and $\pi(f(X)) = \pi(X)$, so (2) gives $a = (a_0, \dots, a_{r-1}) \in K^r$ with $(a, \delta a) \in f(X)$. For $y := a_0$, we have $\nabla^r(y) \in X$.

To see that (3)⇒(4), we use Fact 2.1. Let $r > 0$, $P \in K[X_0, \dots, X_r]$, and $Q \in K[X_0, \dots, X_{r-1}]^{\neq 0}$. Suppose there is $x \in K^{1+r}$ with $P(x) = 0$ and $\frac{\partial P}{\partial X_r}(x) \neq 0$. Then x is a smooth K-rational point of V_P, the zero-locus of P, so we may assume that V_P is smooth and irreducible. Take $X := V_P(K) \setminus V_Q(K)$, so X is éz and $\pi(X) \subseteq K^r$ has $\mathcal{E}_K$-open interior by Fact 2.2(2). Then (3) gives $y \in K$ with $\nabla^r(y) \in X$.

Finally, suppose that (K, δ) is differentially large and let V, X be as in (1). Then τV is smooth as well, so we can use [24, Theorem B(1)] to take smooth irreducible disjoint subvarieties $W_1, \dots, W_n$ of τV and $\mathcal{L}_{\text{ring}}(K)$-definable $\mathcal{E}_K$-open subsets $X_i \subseteq W_i(K)$ for each i with $X = X_1 \cup \dots \cup X_n$. Then $\pi_V(X_i)$ is $\mathcal{E}_K$-open in $V(K)$ for some i, so we set $W := W_i$ for this i and we replace X with X_i. Further shrinking X, we may assume that X is a basic $\mathcal{E}_K$-open subset of $W(K)$, so X is existentially $\mathcal{L}_{\text{ring}}(K)$-definable. As X is $\mathcal{E}_K$-open in $W(K)$, we can take an elementary $\mathcal{L}_{\text{ring}}$-extension $K^* \succcurlyeq K$ containing a tuple $(x, u) \in X^*$ that is K-generic in W using Fact 2.2(2). Then x is K-generic in V and $(x, u) \in \tau V(K^*)$, so we may extend δ to a derivation $\delta^*: K^* \to K^*$ with $\delta^* x = u$; see [11, Chapter IV, Theorems 14 and 18]. As (K, δ) is differentially large and K is $\mathcal{L}_{\text{ring}}$-existentially closed in K^*, the differential field (K, δ) is existentially closed in (K^*, δ^*). As X is existentially $\mathcal{L}_{\text{ring}}(K)$-definable, we find $a \in V(K)$ with $(a, \delta a) \in X$. □

When K is not large, the conditions in Proposition 2.3 are trivially equivalent (they never hold). Using (3)⇒(4) of Proposition 2.3 and Fact 2.2, we have:

Corollary 2.4. *Suppose that K expands a large field and that δ is generic. Then the underlying differential field (K, δ) is differentially large.*

An **éz-field** is by definition a large field for which every definable set is an éz set. For these fields, we get the converse:

Corollary 2.5. *Suppose that $\mathcal{L} = \mathcal{L}_{\text{ring}}$ and that K is an éz-field. If (K, δ) is differentially large, then δ is generic.*

3. Transferring NTP_2 and NATP

In this section, K is an algebraically bounded structure and δ is a generic derivation on K. We also assume that (K, δ) is sufficiently saturated.

A formula $\varphi(x, y)$ (where x, y are tuples of variables) has the **tree property of the second kind** (TP_2) if there is an array of tuples $(a_{i,j})_{i,j<\omega}$ such that

(1) The formula $\varphi(x, a_{i,j}) \wedge \varphi(x, a_{i,j'})$ is inconsistent for all i and all $j < j'$.
(2) The partial type $\{\varphi(x, a_{i,f(i)}) : i < \omega\}$ is consistent for all $f: \omega \to \omega$.

A theory T has TP_2 if some formula has TP_2.

Theorem 3.1. *If T^δ_g has TP_2, then so does T.*

Proof. Assume that T^δ_g has TP_2, as witnessed by an $\mathcal{L}^\delta$-formula $\varphi(x, y)$ and an array of tuples $(a_{i,j})_{i,j<\omega}$. By [4, Lemma 3.2], we may also assume that $|x| = 1$. By Fact 1.4(2), the formula φ is equivalent to a formula of the form $\psi(\nabla^r x, \nabla^s y)$ for natural numbers r, s and an $\mathcal{L}$-formula ψ. By replacing $a_{i,j}$ by $\nabla^s(a_{i,j})$ and augmenting y, we may assume that $s = 1$. For the rest of this proof, we fix r and an $\mathcal{L}$-formula $\psi(x_0, \dots, x_r, y)$ (each x_i unary) such that $\psi(\nabla^r x, y)$ has TP_2. We assume that r is minimal with this property. Note that if $r = 0$, then ψ is an $\mathcal{L}$-formula, so T has TP_2 and we are done. Thus, we assume for the remainder of the proof that $r > 0$. We fix an array $(a_{i,j})_{i,j<\omega}$ witnessing TP_2, and we may arrange that this array is *strongly $\mathcal{L}^\delta$-indiscernible*, meaning that each row $(a_{i,j})_{j<\omega}$ is $\mathcal{L}^\delta$-indiscernible over the other rows and that the sequence of rows is $\mathcal{L}^\delta$-indiscernible.

For $i, j < \omega$, let

$$X_{i,j} \coloneqq \{(x_0, \ldots, x_r) \in K^{r+1} : K \models \psi(x_0, \ldots, x_r, a_{i,j})\}, \qquad X^{\nabla}_{i,j} \coloneqq \{x \in K : \nabla^r x \in X_{i,j}\}.$$

Then $X^{\nabla}_{i,j} \cap X^{\nabla}_{i,j'} = \emptyset$ for all i and $j \neq j'$, but $\bigcap_i X^{\nabla}_{i,f(i)}$ is nonempty for all $f\colon \omega \to \omega$. Let $\pi\colon K^{r+1} \to K^r$ be the projection map onto the first r coordinates.

Claim 1. *The projection $\pi(X_{i,j})$ has dimension r for all i, j.*

Proof of Claim 1. Suppose not, so we find a polynomial $P(x_0, \ldots, x_{r-1}, y)$ such that $P(x_0, \ldots, x_{r-1}, a_{i,j})$ is not identically zero but vanishes on $\pi(X_{i,j})$. In particular, $P(\nabla^{r-1}x, a_{i,j}) = 0$ for all $x \in X^{\nabla}_{i,j}$. This yields a rational function Q such that $\delta^r x = Q(\nabla^{r-1}x, \nabla^r a_{i,j})$ for all $x \in X^{\nabla}_{i,j}$. Thus, $\psi(\nabla^r x, a_{i,j})$ is equivalent to the formula

$$\psi(\nabla^{r-1}x, Q(\nabla^{r-1}x, \nabla^r a_{i,j}), a_{i,j})$$

contradicting minimality of r. □

Claim 2. *Let $f\colon \omega \to \omega$ and let $n > 0$. Then the set*

$$\pi(X_{0,f(0)} \cap X_{1,f(1)} \cap \cdots \cap X_{n-1,f(n-1)})$$

has dimension r.

Proof of Claim 2. For each $i, j < \omega$, set

$$b_{i,j} \coloneqq (a_{ni+k,j+f(k)})_{k<n}, \qquad \theta(x, b_{i,j}) \coloneqq \textstyle\bigwedge_{k<n} \psi(\nabla^r x, a_{ni+k,j+f(k)}).$$

Then the formula θ has TP_2, as witnessed by $(b_{i,j})_{i,j<\omega}$. The claim follows by minimality of r and the previous claim. □

For each i and each $n > 0$, we set

$$F_{i,n} \coloneqq \pi(X_{i,0} \cap X_{i,1} \cap \cdots \cap X_{i,n}),$$

and we let $Z_{i,n}$ denote the Zariski closure of $F_{i,n}$. Then $Z_{0,0} \supseteq Z_{0,1} \supseteq Z_{0,2} \supseteq \cdots$, so Notherianity of the Zariski topology gives m with $Z_{0,m} = Z_{0,n}$ for $n \geqslant m$. Note that $Z_{0,m}$ is then $\mathcal{L}(a_{0,0}, \ldots, a_{0,m})$-definable, and we let $Z_{i,m}$ denote the corresponding $\mathcal{L}(a_{i,0}, \ldots, a_{i,m})$-definable Zariski closed set, so $Z_{i,m} = Z_{i,n}$ for $n \geqslant m$ by indiscernibility. We note that

$$\pi(X_{i,j_0} \cap X_{i,j_1} \cap \cdots \cap X_{i,j_m}) \subseteq Z_{i,m} \tag{3.1}$$

for all i and all $m < j_0 < j_1 < \cdots < j_m$. Indeed, let Z be the Zariski closure of $\pi(X_{i,j_0} \cap X_{i,j_1} \cap \cdots \cap X_{i,j_n})$. If $Z \not\subseteq Z_{i,m}$, then $Z \cap Z_{i,m}$ is a proper Zariski closed subset of Z, and thus of $Z_{i,m}$ as well by indiscernibility of the sequence $(a_{i,j})_{j<\omega}$, but this intersection contains Z_{i,j_m}, contradicting our choice of m.

Note that $\pi(X_{i,1} \cap X_{i,2})$ has dimension $< r$ for each i; if not, then the axioms of T^{δ}_g would give $a \in K$ with $\nabla^r a \in X_{i,1} \cap X_{i,2}$, contradicting that $X^{\nabla}_{i,1} \cap X^{\nabla}_{i,2} = \emptyset$. Thus, $m > 0$ and $Z_{i,m}$ has dimension $< r$ for each i. Now for each i, j, let $c_{i,j} \coloneqq (a_{i,0}, \ldots, a_{i,m}, a_{i,j+m+1})$, and set

$$Y_{i,j} \coloneqq \{(x_0, \ldots, x_r) \in X_{i,j+m+1} : (x_0, \ldots, x_{r-1}) \notin Z_{i,m}\}$$

Then $Y_{i,j}$ is defined by some $\mathcal{L}$-formula $\theta(x_0, \ldots, x_r, c_{i,j})$, and we claim that this formula has $(m+1)$-TP_2, meaning that

$$Y_{i,j_0} \cap \cdots \cap Y_{i,j_m} = \emptyset$$

for all i and all $j_0 < \cdots < j_m$, but that the intersection $\bigcap_i Y_{i,f(i)}$ is nonempty for all $f\colon \omega \to \omega$. The first part follows by (3.1). For the second part, let $f\colon \omega \to \omega$ and let n be given. Then the projection

$$\pi\big(\textstyle\bigcap_{i<n} Y_{i,f(i)}\big) = \pi\big(\bigcap_{i<n} X_{i,f(i)}\big) \setminus \bigcup_{i<n} Z_{i,m}$$

has dimension r by Claim 2. In particular, this intersection is nonempty, so $\bigcap_{i<\omega} Y_{i,f(i)}$ is nonempty as well. By [15, Proposition 5.7], T has TP_2, as witnessed by some finite conjunction of the formula θ. □

This approach works for other tree properties that can be reduced to one variable and witnessed by strongly indiscernible parameters. For completeness, we show how to adapt our approach to the antichain tree property, introduced in [1, Definition 4.1].

A formula $\varphi(x, y)$ has the antichain tree property (ATP) if there exists a tree-indexed set of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ such that

(1) The formula $\varphi(x, a_\eta) \wedge \varphi(x, a_\nu)$ is inconsistent whenever η is a strict truncation of ν.
(2) The partial type $\{\varphi(x, a_\eta) : \eta \in A\}$ is consistent for any antichain $A \subseteq 2^{<\omega}$.

A theory has ATP if there is a formula that has ATP. We say that T has **NATP** if it does not have ATP. Any ATP theory is TP_2 and SOP_1; see [1, Propositions 4.4 and 4.6].

For $\eta, \nu \in 2^{<\omega}$, we write $\eta \triangleleft \nu$ to indicate that η is a strict truncation of ν, and we write $\eta^\frown\nu$ to denote the concatenation of η and ν. Given also $A \subseteq 2^{<\omega}$, we put $\eta^\frown A := \{\eta^\frown\nu : \nu \in A\}$.

Theorem 3.2. *If T_g^δ has ATP, then so does T.*

Proof. Assume that T_g^δ has ATP. Using [2, Fact 2.5 and Theorem 3.17] and arguing as in the proof of Theorem 3.1, we may assume that this is witnessed by a formula $\psi(\nabla^r x, y)$ where x is unary, $\psi(x_0, \ldots, x_r, y)$ is an $\mathcal{L}$-formula, and r is minimal, along with a strongly indiscernible tree-indexed set of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$; see [2, Definition 2.4] for the precise definition of strong indiscernibility. As before, let

$$X_\eta := \{(x_0, \ldots, x_r) \in K^{r+1} : K \models \psi(x_0, \ldots, x_r, a_\eta)\}, \qquad X_\eta^\nabla := \{x \in K : \nabla^r x \in X_\eta\}$$

for $\eta \in 2^{<\omega}$, and let $\pi\colon K^{r+1} \to K^r$ be the projection map onto the first r coordinates. By the proof of Claim 1 above, $\pi(X_\eta)$ has dimension r for all η. Obtaining an analog of Claim 2 takes a bit more work:

Claim. *Let $A \subseteq 2^{<\omega}$ be a finite nonempty antichain. Then the set $\pi\big(\bigcap_{\eta \in A} X_\eta\big)$ has dimension r.*

Proof. Fix $\nu \in A$. For $\eta = \langle i_0, i_1, \ldots, i_{m-1}\rangle \in 2^{<\omega}$, we set

$$\nu^\eta := \nu^\frown\langle i_0\rangle^\frown\nu^\frown\langle i_1\rangle^\frown \cdots ^\frown \nu^\frown\langle i_{m-1}\rangle, \qquad A_\eta := (\nu^\eta)^\frown A.$$

Then $A_\varnothing = A$ and for $B \subseteq 2^{<\omega}$, the set $\bigcup_{\eta \in B} A_\eta$ is an antichain if and only if B is an antichain. Set

$$b_\eta := (a_\mu)_{\mu \in A_\eta}, \qquad \theta(x, b_\eta) = \textstyle\bigwedge_{\mu \in A_\eta} \psi(\nabla^r x, a_\mu).$$

Then the formula θ has ATP, as witnessed by $(b_\eta)_{\eta \in 2^{<\omega}}$. We conclude by minimality of r as above. □

For $n \in \mathbb{N}$, we set $n\langle 0\rangle := \langle 0, 0, \ldots, 0\rangle \in 2^n$ (so $0\langle 0\rangle = \varnothing$). We set $F_n := \pi\big(\bigcap_{i \leqslant n} X_{i\langle 0\rangle}\big)$, we let Z_n denote the Zariski closure of F_n, and we take m with $Z_m = Z_n$ for $n \geqslant m$. Arguing as in the proof of Theorem 3.1, we have that $m > 0$, that $\dim(Z_m) < r$, and that

$$\pi(X_{\eta_0} \cap X_{\eta_1} \cap \cdots \cap X_{\eta_m}) \subseteq Z_m \tag{3.2}$$

for all $m\langle 0\rangle \triangleleft \eta_0 \triangleleft \eta_1 \triangleleft \cdots \triangleleft \eta_m$ (this uses that $(b_\eta)_{\eta \in C}$ and $(b_\nu)_{\nu \in C'}$ have the same $\mathcal{L}^\delta$-type for any two finite chains $C, C' \subseteq 2^{<\omega}$ of the same length, as a consequence of strong indiscernibility). Now for each $\nu \in 2^{<\omega}$, let $c_\nu := (a_\varnothing, \ldots, a_{m\langle 0\rangle}, a_{m\langle 0\rangle^\frown\nu})$ and set

$$Y_\nu := \{(x_0, \ldots, x_r) \in X_{m\langle 0\rangle^\frown\nu} : (x_0, \ldots, x_{r-1}) \notin Z_m\}.$$

Then Y_ν is defined by an $\mathcal{L}$-formula $\theta(x_0, \ldots, x_r, c_\nu)$. Arguing as in the proof of Theorem 3.1, using (3.2) and the Claim, we see that θ has m-ATP, meaning that $\bigcap_{\nu \in A} Y_\nu \neq \emptyset$ for any antichain A, but that $\bigcap_{\nu \in C} Y_\nu = \emptyset$ for any chain C of length m. By [2, Lemma 3.20], T has ATP. □

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References

[1] Jinhoo Ahn and Joonhee Kim, SOP_1, SOP_2*, and antichain tree property*, Ann. Pure Appl. Logic **175** (2024), no. 3, Paper No. 103402, 26.
[2] Jinhoo Ahn, Joonhee Kim, and Junguk Lee, *On the antichain tree property*, J. Math. Log. **23** (2023), no. 2, Paper No. 2250021, 36.
[3] Matthias Aschenbrenner, Artem Chernikov, Allen Gehret, and Martin Ziegler, *Distality in valued fields and related structures*, Trans. Amer. Math. Soc. **375** (2022), no. 7, 4641–4710.
[4] Artem Chernikov, *Theories without the tree property of the second kind*, Ann. Pure Appl. Logic **165** (2014), no. 2, 695–723.
[5] Pablo Cubides Kovacsics and Françoise Point, *Topological fields with a generic derivation*, Ann. Pure Appl. Logic **174** (2023), no. 3, Paper No. 103211.
[6] Arno Fehm, *Subfields of ample fields. Rational maps and definability*, J. Algebra **323** (2010), no. 6, 1738–1744.
[7] Antongiulio Fornasiero and Elliot Kaplan, *Generic derivations on o-minimal structures*, J. Math. Log. **21** (2021), no. 2, Paper No. 2150007, 45.

[8] Antongiulio Fornasiero and Giuseppina Terzo, *Generic derivations on algebraically bounded structures*, J. Symbolic Logic (To appear).
[9] ———, *Generic derivations on algebraically bounded structures II: model theoretic properties*, (2025), Preprint. `https://arxiv.org/abs/2507.22181`.
[10] Nicolas Guzy and Françoise Point, *Topological differential fields*, Ann. Pure Appl. Logic **161** (2010), no. 4, 570–598.
[11] Nathan Jacobson, *Lectures in abstract algebra. Vol III: Theory of fields and Galois theory*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London-New York, 1964.
[12] Will Johnson, Chieu-Minh Tran, Erik Walsberg, and Jinhe Ye, *The étale-open topology and the stable fields conjecture*, J. Eur. Math. Soc. (JEMS) **26** (2024), no. 10, 4033–4070.
[13] Will Johnson and Jinhe Ye, *A note on geometric theories of fields*, Model Theory **2** (2023), no. 1, 121–132.
[14] ———, *Curve-excluding fields*, J. Eur. Math. Soc. (JEMS) (to appear).
[15] Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow, *Tree indiscernibilities, revisited*, Arch. Math. Logic **53** (2014), no. 1-2, 211–232.
[16] Omar León Sánchez and Shezad Mohamed, *Neostability transfers in derivation-like theories*, Model Theory **4** (2025), no. 2, 177–201.
[17] Omar León Sánchez and Marcus Tressl, *Differentially large fields*, Algebra Number Theory **18** (2024), no. 2, 249–280.
[18] ———, *On ordinary differentially large fields*, Canad. J. Math. (to appear).
[19] Rahim Moosa, *Six lectures on model theory and differential-algebraic geometry*, (2022), Preprint, `https://arxiv.org/abs/2210.16684`.
[20] Françoise Point, *Definability of types and VC density in differential topological fields*, Arch. Math. Logic **57** (2018), no. 7-8, 809–828.
[21] Florian Pop, *Little survey on large fields—old & new*, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014, pp. 432–463.
[22] Marcus Tressl, *The uniform companion for large differential fields of characteristic 0*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 3933–3951.
[23] Lou van den Dries, *Dimension of definable sets, algebraic boundedness and Henselian fields*, Ann. Pure Appl. Logic **45** (1989), no. 2, 189–209.
[24] Erik Walsberg and Jinhe Ye, *Éz fields*, J. Algebra **614** (2023), 611–649.

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